A RECURRENCE RELATION FOR THE "INV" ANALOGUE OF q-EULERIAN POLYNOMIALS

CHAK-ON CHOW

ABSTRACT. We study in the present work a recurrence relation, which has long been overlooked, for the q-Eulerian polynomial $A_n^{\mathrm{des,inv}}(t,q) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)}q^{\mathrm{inv}(\sigma)}$, where $\mathrm{des}(\sigma)$ and $\mathrm{inv}(\sigma)$ denote, respectively, the descent number and inversion number of σ in the symmetric group $\mathfrak{S}_n$ of degree n. We give an algebraic proof and a combinatorial proof of the recurrence relation.

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of degree n. Any element σ of $\mathfrak{S}_n$ is represented by the word $\sigma_1\sigma_2\cdots\sigma_n$, where $\sigma_i = \sigma(i)$ for $i = 1, 2, \ldots, n$. Two well-studied statistics on $\mathfrak{S}_n$ are the descent number and the inversion number defined by

$$\mathrm{des}(\sigma) := \sum_{i=1}^{n} \chi(\sigma_i > \sigma_{i+1}),$$
$$\mathrm{inv}(\sigma) := \sum_{1\leqslant i<j\leqslant n} \chi(\sigma_i > \sigma_j),$$

respectively, where $\sigma_{n+1} := 0$ and $\chi(P) = 1$ or 0 depending on whether the statement P is true or not. It is well-known that des is Eulerian and that inv is Mahonian. The generating function of the Euler-Mahonian pair (des, inv) over $\mathfrak{S}_n$ is the following q-Eulerian polynomial:

$$A_n^{\mathrm{des,inv}}(t,q) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)}q^{\mathrm{inv}(\sigma)}.$$

It is clear that $A_n(t,1) \equiv A_n(t)$, the classical Eulerian polynomial. Let z and q be commuting indeterminates. For $n \geqslant 0$, let $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ be a q-integer, and $[n]_q! := [1]_q[2]_q\cdots[n]_q$ be a q-factorial. Define a q-exponential function by

$$e(z;q) := \sum_{n\geqslant 0} \frac{z^n}{[n]_q!}.$$

Stanley [6] proved that

$$A^{\mathrm{des,inv}}(x,t;q) := \sum_{n\geqslant 0} A_n^{\mathrm{des,inv}}(t,q)\frac{x^n}{[n]_q!} = \frac{1-t}{1-te(x(1-t);q)}. \tag{1}$$

2000 *Mathematics Subject Classification.* 05A05, 05A15.

Key words and phrases. Permutations, descents, inversions, Eulerian polynomials, recurrence relation.

Alternate proofs of (1) have also been given by Garsia [4] and Gessel [5]. Désarménien and Foata [2] observed that the right side of (1) is precisely

$$\left(1 - t\sum_{n\geqslant 1}(1-t)^{n-1}\frac{x^n}{[n]_q!}\right)^{-1},$$

and from which they obtained a "semi" q-recurrence relation for $A_n^{\mathrm{des,inv}}(t,q)$, namely,

$$A_n^{\mathrm{des,inv}}(t,q) = t(1-t)^{n-1} + \sum_{1\leqslant i\leqslant n-1}\begin{bmatrix} n \\ i \end{bmatrix}_q A_i^{\mathrm{des,inv}}(t,q)t(1-t)^{n-1-i}.$$

The above q-recurrence relation is "semi" in the sense that the summands on the right involve two factors one of which depends on q whereas the other does not. We shall establish in the present note that a "fully" q-recurrence relation for $A_n^{\mathrm{des,inv}}(t,q)$ exists such that both factors of the summands depend on q (see Theorem 2.2 below). In the next section, we derive this recurrence relation algebraically. In the final section, we give a combinatorial proof of this recurrence relation.

2. The recurrence relation

We derive in the present section the recurrence relation by algebraic means.

Let $\mathbb{Q}$ denote, as customary, the set of rational numbers. Let x be an indeterminate, $\mathbb{Q}[x]$ be the ring of polynomials in x over $\mathbb{Q}$, and $\mathbb{Q}[[x]]$ the ring of formal power series in x over $\mathbb{Q}$. We introduce an Eulerian differential operator δ_x in x by

$$\delta_x(f(x)) = \frac{f(qx)-f(x)}{qx-x},$$

for any $f(x) \in \mathbb{Q}[q][[x]]$ in the ring of formal power series in x over $\mathbb{Q}[q]$. It is easy to see that

$$\delta_x(x^n) = [n]_q x^{n-1},$$

so that as $q \to 1$, $\delta_x(x^n) \to nx^{n-1}$, the usual derivative of x^n. See [1] for further properties of δ_x.

Lemma 2.1. *We have* $\delta_x(e(x(1-t);q) = (1-t)e(x(1-t);q)$.

Proof. This follows from

$$\begin{aligned}\delta_x(e(x(1-t);q) &= \frac{e(qx(1-t);q) - e(x(1-t);q)}{(q-1)x} \\ &= \sum_{n\geqslant 0}\frac{q^n x^n(1-t)^n - x^n(1-t)^n}{(q-1)x[n]_q!} \\ &= \sum_{n\geqslant 1}\frac{x^{n-1}(1-t)^n}{[n-1]_q!} \\ &= (1-t)e(x(1-t);q).\end{aligned}$$

□

THEOREM 2.2. *For $n \geqslant 1$, $A_n^{\mathrm{des,inv}}(t,q)$ satisfies*

$$A_{n+1}^{\mathrm{des,inv}}(t,q) = (1+tq^n)A_n^{\mathrm{des,inv}}(t,q) + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_{n-k}^{\mathrm{des,inv}}(t,q) A_k^{\mathrm{des,inv}}(t,q). \tag{2}$$

Proof. From (1) we have that

$$te(x(1-t);q) = \frac{A^{\mathrm{des,inv}}(x,t;q) - (1-t)}{A^{\mathrm{des,inv}}(x,t;q)}. \tag{3}$$

Applying δ_x to both sides of (1), and using Lemma 2.1, (1) and (3), we have

$$\begin{aligned}
\sum_{n\geqslant 0} A_{n+1}^{\mathrm{des,inv}}(t,q)\frac{x^n}{[n]_q!} &= \frac{(1-t)}{(q-1)x}\left(\frac{1}{1-te(qx(1-t);q)} - \frac{1}{1-te(x(1-t);q)}\right) \\
&= \frac{t(1-t)\delta_x(e(x(1-t);q)}{[1-te(x(1-t);q)][1-te(qx(1-t);q)]} \\
&= \frac{t(1-t)^2 e(x(1-t);q)}{[1-te(x(1-t);q)][1-te(qx(1-t);q)]} \\
&= [A^{\mathrm{des,inv}}(x,t;q) - (1-t)]A^{\mathrm{des,inv}}(qx,t;q).
\end{aligned}$$

Extracting the coefficients of x^n, we finally have

$$\begin{aligned}
A_{n+1}^{\mathrm{des,inv}}(t,q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_{n-k}^{\mathrm{des,inv}}(t,q) A_k^{\mathrm{des,inv}}(t,q) - (1-t)q^n A_n^{\mathrm{des,inv}}(t,q) \\
&= (1+tq^n)A_n^{\mathrm{des,inv}}(t,q) + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_{n-k}^{\mathrm{des,inv}}(t,q) A_k^{\mathrm{des,inv}}(t,q).
\end{aligned}$$

□

The identity (2) is a q-analogue of the following convolution-type recurrence [3, p. 70]

$$A_{n+1}(t) = (1+t)A_n(t) + \sum_{k=1}^{n-1} \binom{n}{k} A_{n-k}(t)A_k(t),$$

satisfied by the classical Eulerian polynomials $A_n(t) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)}$.

3. A combinatorial proof

We give a combinatorial proof of Theorem 2.2 in the present section.

Recall that elements of $\mathfrak{S}_{n+1}$ can be obtained by inserting $n+1$ to elements of $\mathfrak{S}_n$. Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. Denote by $\sigma_{+k} = \sigma_1 \cdots \sigma_k (n+1) \sigma_{k+1} \cdots \sigma_n$, $0 \leqslant k \leqslant n$. It is easy to see that

$$\begin{aligned}
&\mathrm{des}(\sigma_{+0}) = \mathrm{des}(\sigma) + 1, \quad \mathrm{inv}(\sigma_{+0}) = \mathrm{inv}(\sigma) + n, \\
&\mathrm{des}(\sigma_{+n}) = \mathrm{des}(\sigma), \quad \mathrm{inv}(\sigma_{+n}) = \mathrm{inv}(\sigma),
\end{aligned}$$

and for $1 \leqslant k \leqslant n-1$,

$$\begin{aligned}
\mathrm{des}(\sigma_{+k}) &= \mathrm{des}(\sigma_1\cdots\sigma_k) + \mathrm{des}(\sigma_{k+1}\cdots\sigma_n),\\
\mathrm{inv}(\sigma_{+k}) &= \mathrm{inv}(\sigma_1\cdots\sigma_k) + \mathrm{inv}(\sigma_{k+1}\cdots\sigma_n)\\
&\qquad + n-k+\#\{(r,s)\colon \sigma_r>\sigma_s, 1\leqslant r\leqslant k, k+1\leqslant s\leqslant n\}.
\end{aligned}$$

Let $S=\{\sigma_1,\ldots,\sigma_k\}$. Then the partial permutations $\sigma_1\cdots\sigma_k \in \mathfrak{S}(S)$ and $\sigma_{k+1}\cdots\sigma_n \in \mathfrak{S}([n]\setminus S)$, where $\mathfrak{S}(S)$ denotes the group of permutations of the set S. It is clear that the product $\mathfrak{S}(S)\times\mathfrak{S}([n]\setminus S)$ is a subgroup of $\mathfrak{S}_n$ isomorphic to $\mathfrak{S}_k\times\mathfrak{S}_{n-k}$. Also, the quotient $\mathfrak{S}_n/(\mathfrak{S}_k\times\mathfrak{S}_{n-k})\cong\binom{[n]}{k}$ (see [8, p. 351]), where $\binom{[n]}{k}$ denotes the set of all k-subsets of $[n]$, which is in bijective correspondence with the set of multipermutations $\mathfrak{S}(\{1^k,2^{n-k}\})$ of the multiset $\{1^k,2^{n-k}\}$ consisting of k copies of 1's and $n-k$ copies of 2's.

Define a multipermutation $w=w_1w_2\cdots w_n\in\mathfrak{S}(\{1^k,2^{n-k}\})$ by

$$w_i=\begin{cases}1 & \text{if } i\in S=\{\sigma_1,\ldots,\sigma_k\},\\ 2 & \text{if } i\in[n]\setminus S=\{\sigma_{k+1},\ldots,\sigma_n\}.\end{cases}$$

Let $1\leqslant i<j\leqslant n$. It is clear that (i,j) is an inversion of w if and only if $i=\sigma_s$, $j=\sigma_r$ for some $1\leqslant r\leqslant k$, $k+1\leqslant s\leqslant n$ and $\sigma_r>\sigma_s$, so that

$$\#\{(r,s)\colon \sigma_r>\sigma_s, 1\leqslant r\leqslant k, k+1\leqslant s\leqslant n\}=\mathrm{inv}(w).$$

As S ranges over $\binom{[n]}{k}$, w so defined ranges over $\mathfrak{S}(\{1^k,2^{n-k}\})$. Putting pieces together and using the fact [7, Proposition 1.3.17] that

$$\sum_{w\in\mathfrak{S}(\{1^k,2^{n-k}\})} q^{\mathrm{inv}(w)}=\begin{bmatrix}n\\k\end{bmatrix}_q,$$

we have

(4)

$$\begin{aligned}
&A_{n+1}^{\mathrm{des,inv}}(t,q)\\
&=\sum_{k=0}^{n}\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma_{+k})}q^{\mathrm{inv}(\sigma_{+k})}\\
&=(1+tq^n)A_n^{\mathrm{des,inv}}(t,q)+\sum_{k=1}^{n-1}\sum_{\substack{\sigma_1\cdots\sigma_k\in\mathfrak{S}_k\\ \sigma_{k+1}\cdots\sigma_n\in\mathfrak{S}_{n-k}\\ w\in\mathfrak{S}(\{1^k,2^{n-k}\})}} t^{\mathrm{des}(\sigma_1\cdots\sigma_k)+\mathrm{des}(\sigma_{k+1}\cdots\sigma_n)}q^{\mathrm{inv}(\sigma_1\cdots\sigma_k)+\mathrm{inv}(\sigma_{k+1}\cdots\sigma_n)+n-k+\mathrm{inv}(w)}\\
&=(1+tq^n)A_n^{\mathrm{des,inv}}(t,q)+\sum_{k=1}^{n-1}q^{n-k}\sum_{w\in\mathfrak{S}(\{1^k,2^{n-k}\})}q^{\mathrm{inv}(w)}\sum_{\tau\in\mathfrak{S}_k}t^{\mathrm{des}(\tau)}q^{\mathrm{inv}(\tau)}\sum_{\pi\in\mathfrak{S}_{n-k}}t^{\mathrm{des}(\pi)}q^{\mathrm{inv}(\pi)}\\
&=(1+tq^n)A_n^{\mathrm{des,inv}}(t,q)+\sum_{k=1}^{n-1}q^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q A_k^{\mathrm{des,inv}}(t,q)A_{n-k}^{\mathrm{des,inv}}(t,q),
\end{aligned}$$

which is equivalent to (2) (by virtue of the symmetry of the q-binomial coefficient).

References

[1] G.E. Andrews, On the foundations of combinatorial theory V, Eulerian differential operators, *Stud. Appl. Math.* **50** (1971), 345–375.
[2] J. Désarménien and D. Foata, Signed Eulerian numbers, *Discrete Math.* **99** (1992), 49–58.
[3] D. Foata and M.-P. Schützenberger, Théorie géométrique des polynômes Eulériens, *Lecture Notes in Mathematics*, vol. 138, Springer-Verlag, Berlin-New York, 1970.
[4] A.M. Garsia, On the "maj" and "inv" analogues of Eulerian polynomials, *Linear and Multilinear Algebra* **8** (1979), 21–34.
[5] I.M. Gessel, *Generating Functions and Enumeration of Sequences*, Ph.D. thesis, Massachusetts Institute of Technology, June 1977.
[6] R.P. Stanley, Binomial posets, Möbius inversion and permutation enumeration, *J. Combin. Theory Ser. A* **20** (1976), 336–356.
[7] R.P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, Cambridge, 1997.
[8] R.P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, 1999.

Department of Mathematics and Information Technology, Hong Kong Institute of Education, 10 Lo Ping Road, Tai Po, New Territories, Hong Kong
E-mail address: cchow@alum.mit.edu