CHARACTERIZATIONS OF BMO BY A_p WEIGHTS AND p-CONVEXITY

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Abstract. We show that the Lebesgue spaces for defining BMO can be replaced by p-convex rearrangement-invariant quasi-Banach function spaces associated with A_p-weighted measures.

1. Introduction

In this paper, we apply the notion of p-convexity to study the characterizations of BMO by rearrangement-invariant quasi-Banach function spaces (r.-i.q-B.f.s) on $(\mathbb{R}^n, \omega)$ where $\omega \in A_\infty$.

The notion of p-convexity $1 \le p \le \infty$ for Banach lattices was introduced in [2, 4, 10]. For the extension of the notion of p-convexity to quasi-Banach space, the reader is referred to [3] p.156.

The notion of p-convexity was used to study the isomorphic properties of Banach lattices (see [11] Volume II, Section 1.d). In this paper, we find that BMO can be characterized by an r.-i.q-B.f.s. on $(\mathbb{R}^n, \omega)$, $\omega \in A_p$, if it is p-convex.

On one hand, this paper shows that p-convexity is not only an abstract notion arising from the Banach space geometry, it also has applications on the study of some concrete function spaces. On the other hand, this paper generalizes the characterizations of BMO by replacing the Lebesgue spaces L^p, $1 \le p < \infty$, by rearrangement-invariant quasi-Banach function spaces.

We recall some results on the characterizations of BMO. Let $\mathcal{B} = \{B(x_0, r) : x_0 \in \mathbb{R}^n, r > 0\}$ where $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$. Let us denote the center and the radius of $B \in \mathcal{B}$ by c_B and r_B, respectively. Let $\mathcal{M}_0$ be the set of locally integrable function on $\mathbb{R}^n$.

Recall that a locally integrable function f belongs to BMO if

$$\|f\|_{BMO} = \sup_{B \in \mathcal{B}} \frac{\|(f - f_B)\chi_B\|_{L^1}}{\|\chi_B\|_{L^1}} < \infty$$

where $f_B = \frac{1}{|B|} \int_B f(x)dx$.

Date: July 30, 2010.

Key words and phrases. BMO, p-convexity, A_p weight, rearrangement-invariant quasi-Banach function space.

More generally, BMO can be defined via the L^p norm. That is,

$$BMO = \left\{ f \in \mathcal{M}_0 : \sup_{B\in\mathcal{B}} \frac{\|(f-f_B)\chi_B\|_{L^p}}{\|\chi_B\|_{L^p}} < \infty \right\}.$$

Thus, the characterization of BMO can be generalized by examining whether we have the following identification

$$BMO = \left\{ f \in \mathcal{M}_0 : \sup_{B\in\mathcal{B}} \frac{\|(f-f_B)\chi_B\|_{L^p(\omega)}}{\|\chi_B\|_{L^p(\omega)}} < \infty \right\} \tag{1.1}$$

where $\omega \in A_\infty$.

As Lemma 3.1 shows, one inclusion is easy to obtain. One of the main result for this paper is that the identity (1.1) is true provided that $\omega \in A_p$, see Theorem 3.3.

We further extend our result by considering the r.-i.q-B.f.s. Y_ω on $(\mathbb{R}^n, \omega)$ where $\omega \in A_\infty$. That is, we investigate whether the characterization

$$BMO = \left\{ f \in \mathcal{M}_0 : \sup_{B\in\mathcal{B}} \frac{\|(f-f_B)\chi_B\|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}} < \infty \right\} \tag{1.2}$$

is valid. This result is presented in Theorem 3.9.

For any r.-i.q-B.f.s Y_ω on $(\mathbb{R}^n, \omega)$, we introduce BMO_{Y_ω}. It is defined by

$$BMO_{Y_\omega} = \left\{ f \in \mathcal{M}_0 : \sup_{B\in\mathcal{B}} \frac{\|(f-f_B)\chi_B\|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}} < \infty \right\}.$$

Write $\|f\|_{BMO_{Y_\omega}} = \sup_{B\in\mathcal{B}} \dfrac{\|(f-f_B)\chi_B\|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}}$.

To prove the embedding $BMO \hookrightarrow BMO_{Y_\omega}$, we establish the John-Nirenberg inequality for r.-i.q-B.f.s on $(\mathbb{R}^n, \omega)$, see Proposition 3.5. In fact, with that proposition, the embedding

$$BMO \hookrightarrow BMO_{Y_\omega} \tag{1.3}$$

holds for r.-i.q-B.f.s Y_ω on $(\mathbb{R}^n, \omega)$, $\omega \in A_\infty$.

The unweighted version of the characterization (1.2) is presented in [7]. For the unweighted case, an expected condition on the Boyd indices of the rearrangement-invariant Banach function space (r.-i.B.f.s.) X on $(\mathbb{R}^n, |\cdot|)$ where $|\cdot|$ is the Lebesgue measure is enough to guarantee the characterization of BMO by X. However, when Y_ω is an r.-i.q-B.f.s. on $(\mathbb{R}^n, \omega)$ and $\omega \in A_p$, we need the notion of p-convexity to establish the reverse inequality of (1.3).

Moreover, for a general r.-i.q-B.f.s. Y_ω on $(\mathbb{R}^n, \omega)$, it is not necessarily a subset of $\mathcal{M}_0$. That is, f_B is not necessarily well defined for any $B \in \mathcal{B}$ and $f \in Y_\omega$.

In Theorem 3.9, we figure out a condition imposed on an r.-i.q-B.f.s. so that on one hand, it is a subset of $\mathcal{M}_0$ and on the other hand, the characterization (1.2) is valid. More precisely, we find that if Y_ω is p-convex and its Boyd's indices satisfying $p \le p_{Y_\omega} \le q_{Y_\omega} < \infty$, then $Y_\omega \subset \mathcal{M}_0$ and we have the characterization (1.2).

2. Background materials

We present the notations and terminologies used in this paper.

Even though the A_p class is well-known, for completeness, we offer the definition of A_p weight functions.

Definition 2.1. For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_p weight if

$$\sup_{B \in \mathcal{B}} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C\omega(x), \quad a.e.\, x \in B$$

for some constant $C > 0$. We define $A_\infty = \cup_{p \geq 1} A_p$.

For any $\omega \in A_\infty$ and any Lebesgue measurable set E, write $\omega(E) = \int_E \omega(x) dx$. We have the following characterization of A_∞ weight (see [6] Theorem 9.3.3 (d)).

Theorem 2.1. *A locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ belongs to A_∞ if and only if there exist an $\epsilon > 0$ and a constant $C_0 > 0$ such that for any $B \in \mathcal{B}$ and all measurable subsets E of B, we have*

$$\frac{\omega(E)}{\omega(B)} \leq C_0 \left(\frac{|E|}{|B|} \right)^{\epsilon}. \tag{2.1}$$

We recall the John-Nirenberg inequality in the next theorem (see [6], Theorem 7.1.6).

Theorem 2.2. *There exist constants $C_1, C_2 > 0$ such that for any $\gamma > 0$ and any $B \in \mathcal{B}$,*

$$|\{x \in B : |f(x) - f_B| > \gamma\}| \leq C_1 e^{-\frac{C_2 \gamma}{\|f\|_{BMO}}} |B|, \quad f \in BMO \backslash \mathcal{C}$$

where $\mathcal{C}$ denotes the set of constant functions.

We state some background materials for rearrangement-invariant quasi-Banach function spaces.

Let $\omega \in A_\infty$. For any Lebesgue measurable function f, denote its decreasing rearrangement with respect to $(\mathbb{R}^n, \omega)$ by $f^{*,\omega}$.

We recall the definition of rearrangement-invariant Banach function space from [1, Chapter 1, Definitions 1.1 and 1.3, and Chapter 2, Definition 4.1]. For any $\omega \in A_\infty$, let $\mathcal{M}_\omega$ denote the class of ω-measurable functions.

Definition 2.2. Let $\omega \in A_\infty$. A mapping $\rho : \mathcal{M}_\omega \to [0, \infty]$ is said to be a rearrangement-invariant Banach function norm if for all ω-measurable functions $f, g, \{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}^n$ and $a > 0$, we have

(1) $\rho(f) = 0 \Leftrightarrow f = 0$ ω-a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$
(2) $0 \leq g \leq f$ ω-a.e. $\Rightarrow \rho(g) \leq \rho(f)$
(3) $0 \leq f_n \uparrow f$ ω-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$
(4) $\omega(E) < \infty \Rightarrow \rho(\chi_E) < \infty$
(5) $\omega(E) < \infty \Rightarrow \int_E f(x)\omega(x)dx \leq C_E\rho(f)$ for some $C_E > 0$.
(6) $\rho(f) = \rho(g)$ for every pair of equimeasurable functions f, g.

The collection Y_ω of all functions f in $\mathcal{M}_\omega$ for which $\rho(|f|) < \infty$ is called a *rearrangement-invariant Banach function space* (r.-i.B.f.s.). The norm of Y_ω is given by $\|\cdot\|_{Y_\omega} = \rho(|\cdot|)$.

For any r.-i.B.f.s. Y_ω, according to the Luxemburg representation theorem (see [1] Chapter 2, Theorem 4.10), we have a norm $\rho_{Y_\omega} : \mathcal{M}([0, \infty)) \to [0, \infty]$ where $\mathcal{M}([0, \infty))$ is the set of Lebesgue measurable functions on $[0, \infty)$ such that

$$\|f\|_{Y_\omega} = \rho_{Y_\omega}(f^{*,\omega}).$$

We find that this property is crucial on the definition of Boyd's indices. On the other hand, the validity of the Luxemburg representation theorem relies on the fact that the associated space of an r.-i.B.f.s. is not trivial. But for a general quasi-Banach function space, the associated space may be trivial. That is, item (5) of Definition 2.2 does not necessarily hold for a general quasi-Banach function space. For instance, when $0 < p < 1$, the associated space of $L^p(\mathbb{R}^n)$ is equal to $\{0\}$.

Therefore, we use the subsequent definition for rearrangement-invariant quasi-Banach function spaces.

Definition 2.3. Let $\omega \in A_\infty$. A quasi-Banach function space Y_ω on $(\mathbb{R}^n, \omega)$ is rearrangement-invariant (r.-i.) if

(1) $\|\cdot\|_{Y_\omega}$ is a quasi-norm ;
(2) $\|\cdot\|_{Y_\omega}$ satisfy item (2)-(4) in Definition 2.2;
(3) there exists a quasi-norm $\rho_{Y_\omega} : \mathcal{M}([0, \infty)) \to [0, \infty]$ such that

$$\|f\|_{Y_\omega} = \rho_{Y_\omega}(f^{*,\omega}).$$

We combine the definition of Boyd indices for r.-i.B.f.s. from [1], Chapter 3, Definition 5.10 and the definition of Boyd indices for r.-i.q-B.f.s. from [12] to give the following definition.

Definition 2.4. For each $t > 0$ and any Lebesgue measurable function f on $[0, \infty)$, let E_t denote the dilation operator defined by

$$(E_t f)(x) = f(tx), \quad x \geq 0.$$

The Boyd indices of an r.-i.q-B.f.s. Y_ω are the numbers defined by

$$p_{Y_\omega} = \sup\{p : \exists C > 0, \forall t < 1, \rho_{Y_\omega}(E_t f) \leq Ct^{-\frac{1}{p}}\rho_{Y_\omega}(f)\},$$

$$q_{Y_\omega} = \inf\{q : \exists C > 0, \forall t > 1, \rho_{Y_\omega}(E_t f) \leq Ct^{-\frac{1}{q}}\rho_{Y_\omega}(f)\}.$$

We have $0 \leq p_{Y_\omega} \leq q_{Y_\omega} \leq \infty$. For any quasi-Banach function space Y_ω, let Y'_ω be the associated space (the Köthe dual) of Y_ω (see [13] p.35).

Lemma 2.3. *Let Y_ω be an r.-i.B.f.s. on $(\mathbb{R}^n, \omega)$. For any Lebesgue measurable set E with $\omega(E) < \infty$, we have*

$$\|\chi_E\|_{Y_\omega}\|\chi_E\|_{Y'_\omega} = \omega(E). \tag{2.2}$$

The above lemma is crucial to establish the main result in [7]. The proof of the above lemma is given in [1] Chapter 2, Theorem 5.2.

The identification $BMO_{L^p(\omega)} = BMO$ is valid provided that $\omega \in A_p$. To apply this result to r.-i.q-B.f.s. on $(\mathbb{R}^n, \omega)$, we introduce the notion of p-th power ($1/p$-convexification). For any $0 < p < \infty$ and any quasi-Banach function space Y_ω on $(\mathbb{R}^n, \omega)$, define the p-th power of Y_ω, Y_ω^p by

$$f \in Y_\omega^p \Leftrightarrow |f|^{\frac{1}{p}} \in Y_\omega,$$

and the quasi-norm of Y_ω^p is defined by $\|f\|_{Y_\omega^p} = \||f|^{\frac{1}{p}}\|_{Y_\omega}^p$. The reader is referred to [13] Section 2.2 for a complete discussion on the notion of p-th power of quasi-Banach function space. For $0 < p < 1$, Y_ω^p is a Banach space (see [5] Proposition 1.11) while for $1 < p < \infty$, it is a quasi-Banach space (see [13] Chapter 2, Proposition 2.22).

As claimed on the introduction, conditions on the Boyd indices are not sufficient to assert the characterization of BMO by A_p weights. We need another notion from the geometry of quasi-Banach space. Let $0 < p < \infty$. A quasi-Banach function space X is said to be p-convex if there exists a constant $C > 0$ such that

$$\Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{\frac{1}{p}}\Big\|_X \leq C\Big(\sum_{i=1}^n \|f_i\|_X^p\Big)^{\frac{1}{p}}$$

for any $\{f_i\}_{i=1}^n \subset X$.

Any p-convex quasi-Banach function space is also r-convex provided that $0 < r \leq p$ (see [3] Lemma 4).

The following proposition gives a procedure to obtain an equivalent norm for a p-convex quasi-Banach function space. That procedure was already presented in [4, 10] for Banach lattices.

Proposition 2.4. *Let $1 \leq p < \infty$. If the quasi-Banach function space Y_ω is p-convex, then*

$$\eta_{[p]}(f) = \inf\{\sum_{i=1}^n \|f_i\|_{Y_\omega^p} : |f| \leq \sum_{i=1}^n |f_i|,\ f_i \in Y_\omega^p,\ i = 1, 2, \ldots, n,\ n \in \mathbb{N}\} \tag{2.3}$$

is a lattice norm and is equivalent to $\|\cdot\|_{Y_\omega^p}$. Hence, Y_ω is normable and admits

$$\eta(f) = \big(\eta_{[p]}(|f|^p)\big)^{\frac{1}{p}}$$

as an equivalent lattice norm.

The proof of the above proposition is given by [13, Proposition 2.23].

3. The characterizations of BMO

We present several embedding and characterizations of BMO in this section. We first pay our attention to the weighted Lebesgue space $L^p(\omega)$, $1 \le p < \infty$, $\omega \in A_\infty$.

Lemma 3.1. *Let $0 < p < \infty$ and $\omega \in A_\infty$. We have*

$$BMO \hookrightarrow BMO_{L^p(\omega)}. \tag{3.1}$$

Proof: According to the John-Nirenberg inequality, we have

$$|\{x \in B : |f(x) - f_B| > \gamma\}| \le C_1 e^{-\frac{C_2\gamma}{\|f\|_{BMO}}}|B|.$$

Applying inequality (2.1), we find that

$$\omega(\{x \in B : |f(x) - f_B| > \gamma\}) \le C_0 C_1^\epsilon e^{-\frac{C_2\epsilon\gamma}{\|f\|_{BMO}}}\omega(B) \tag{3.2}$$

for some $C_0 > 0$. Hence, there exists constant $C_3 > 0$ so that for any $B \in \mathcal{B}$,

$$\begin{aligned}\frac{1}{\omega(B)}\|(f - f_B)\chi_B\|^p_{L^p(\omega)} &= \frac{p}{\omega(B)}\int_0^\infty \gamma^{p-1}\omega(\{x \in B : |f(x) - f_B| > \gamma\})d\gamma \\ &\le C_0 C_1^\epsilon p \int_0^\infty \gamma^{p-1} e^{-\frac{C_2\epsilon\gamma}{\|f\|_{BMO}}}d\gamma \le C_3\|f\|^p_{BMO}.\end{aligned}$$

Therefore, the embedding (3.1) is valid. ∎

Lemma 3.2. *If $1 \le p < \infty$ and $\omega \in A_p$, then*

$$BMO_{L^p(\omega)} \hookrightarrow BMO. \tag{3.3}$$

Proof: When $1 < p$, by the Hölder inequality, we obtain

$$\int_B |f(x) - f_B|dx \le \left(\int_B |f(x) - f_B|^p\omega(x)dx\right)^{\frac{1}{p}}\left(\int_B \omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{1}{p'}}.$$

Therefore, the A_p condition concludes that

$$\int_B |f(x) - f_B|dx \le C\|(f - f_B)\chi_B\|_{L^p(\omega)}\frac{|B|}{\omega(B)^{\frac{1}{p}}}.$$

Similarly, the proof of the embedding (3.3) for $p = 1$ follows from the definition of A_1 weight functions. ∎

The above lemmas offer a new characterization for BMO by $L^p(\omega)$, $\omega \in A_p$. We will generalize this characterization of BMO to r.-i.q-B.f.s. on the rest of this section. Even though the characterization of BMO by $L^p(\omega)$ is a special case of the following results, the proof for the general case is, in fact, based on this special case.

Theorem 3.3. *Let $1 \le p < \infty$ and $\omega \in A_p$. We have*

$$BMO = BMO_{L^p(\omega)}.$$

The norms are mutually equivalent.

We now turn our attention to the r.-i.q-B.f.s. on $(\mathbb{R}^n,\omega)$, $\omega\in A_\infty$.

We prove the embedding $BMO \hookrightarrow BMO_{Y_\omega}$ by using the John-Nirenberg inequality. We have a supporting lemma for establishing the John-Nirenberg inequality on Y_ω.

Lemma 3.4. *Let $\omega\in A_\infty$. If Y_ω is an r.-i.q-B.f.s. on $(\mathbb{R}^n,\omega)$ with $q_{Y_\omega}<\infty$, then for any $q>q_{Y_\omega}$, there exists a constant $C>0$ such that for any $x_0\in\mathbb{R}^n$ and $R>r>0$, we have*

$$\frac{\|\chi_{B(x_0,r)}\|_{Y_\omega}}{\|\chi_{B(x_0,R)}\|_{Y_\omega}}\le C\left(\frac{\omega(B(x_0,r))}{\omega(B(x_0,R))}\right)^{\frac{1}{q}}. \tag{3.4}$$

The proof of Lemma 3.4 follows from the definition of Boyd's indices and the facts that

$$\frac{\|\chi_{B(x_0,r)}\|_{Y_\omega}}{\|\chi_{B(x_0,R)}\|_{Y_\omega}}=\frac{\rho_{Y_\omega}(\chi_{[0,\omega(B(x_0,r))]})}{\rho_{Y_\omega}(\chi_{[0,\omega(B(x_0,R))]})}$$

and $E_t\chi_{[0,\omega(B(x_0,R))]}=\chi_{[0,\omega(B(x_0,r))]}$ where $t=\frac{\omega(B(x_0,R))}{\omega(B(x_0,r))}$.

We have the following John-Nirenberg inequality for r.-i.q-B.f.s..

Proposition 3.5. *Let $\omega\in A_\infty$. If Y_ω is an r.-i.q-B.f.s. on $(\mathbb{R}^n,\omega)$ with $q_{Y_\omega}<\infty$, then there exist $K_1,K_2>0$ such that for any $\gamma>0$ and any $B\in\mathcal{B}$,*

$$\|\chi_{\{x\in B:|f(x)-f_B|>\gamma\}}\|_{Y_\omega}\le K_1 e^{-\frac{K_2\gamma}{\|f\|_{BMO}}}\|\chi_B\|_{Y_\omega},\quad \forall f\in BMO\backslash\mathcal{C}.$$

Proof: As $\omega\in A_\infty=\cup_{1\le p<\infty}A_p$, we have $C_4,\delta>0$ (see [6] Corollary 9.3.4 and Proposition 9.1.5.(9)) such that for any $\lambda>1$,

$$\omega(B(x_0,\lambda r))\le C_4\lambda^\delta\omega(B(x_0,r)),\quad \forall x_0\in\mathbb{R}^n.$$

It suffices to consider the case when γ is large, so, without loss of generality, we assume that $(C_4C_0)^{\frac{1}{\delta}}C_1^{\frac{\epsilon}{\delta}}e^{-\frac{C_2\epsilon\gamma}{\delta\|f\|_{BMO}}}<1$.

According to the John-Nirenberg inequality for ω (see (3.2)), we find that

$$\omega(\{x\in B:|f(x)-f_B|>\gamma\})\le\omega(\tilde{B})$$

where $\tilde{B}\in\mathcal{B}$ with

$$c_{\tilde{B}}=c_B\quad\text{and}\quad r_{\tilde{B}}=(C_4C_0)^{\frac{1}{\delta}}C_1^{\frac{\epsilon}{\delta}}e^{-\frac{C_2\epsilon\gamma}{\delta\|f\|_{BMO}}}r_B.$$

As Y_ω is r.-i. with respect to ω (see [1] Chapter 2, Definition 5.1 and Corollary 5.3), we assert that

$$\|\chi_{\{x\in B:|f(x)-f_B|>\gamma\}}\|_{Y_\omega}\le\|\chi_{\tilde{B}}\|_{Y_\omega}.$$

In view of $q_{Y_\omega}<\infty$, (2.1) and Lemma 3.4 guarantee that for any $q>q_{Y_\omega}$

$$\|\chi_{\{x\in B:|f(x)-f_B|>\gamma\}}\|_{Y_\omega}\le K_1e^{\frac{-K_2\gamma}{\|f\|_{BMO}}}\|\chi_B\|_{Y_\omega}$$

where $K_1=(C_4C_0)^{\frac{n\epsilon}{q\delta}}C_1^{\frac{n\epsilon^2}{q\delta}}$ and $K_2=\frac{C_2n\epsilon^2}{q\delta}$. ∎

Theorem 3.6. *Let $\omega \in A_\infty$. Suppose that Y_ω is an r.-i.q-B.f.s. on $(\mathbb{R}^n, \omega)$ with $q_{Y_\omega} < \infty$. Then, we have the embedding*

$$BMO \hookrightarrow BMO_{Y_\omega}. \tag{3.5}$$

Proof: Let κ be the Aoki-Rolewicz index for the quasi-Banach function space Y_ω (see [9] Theorem 1.3). That is, κ is a number such that $\|\cdot\|_{Y_\omega}^\kappa$ is sub-additive on Y_ω. For any $j \in \mathbb{N}$, Proposition 3.5 gives

$$\|\chi_{\{x\in B:2^j<|f(x)-f_B|\le 2^{j+1}\}}\|_{Y_\omega}^\kappa \le K_1^\kappa e^{\frac{-\kappa K_2 2^j}{\|f\|_{BMO}}}\|\chi_B\|_{Y_\omega}^\kappa$$

because $q_{Y_\omega} < \infty$. Multiplying $2^{(j+1)\kappa}$ on both sides and summing over j, we find that

$$\|(f-f_B)\chi_B\|_{Y_\omega}^\kappa \le C\|f\|_{BMO}^\kappa\|\chi_B\|_{Y_\omega}^\kappa \tag{3.6}$$

for some constant $C > 0$. More precisely, we have the above inequality because

$$\sum_{j\in\mathbb{N}} 2^{j\kappa} e^{\frac{-\kappa K_2 2^j}{\|f\|_{BMO}}} \le C\int_0^\infty s^{\kappa-1} e^{\frac{-\kappa K_2 s}{\|f\|_{BMO}}} ds \le C\|f\|_{BMO}^\kappa$$

for some constant $C > 0$ independent of $f \in BMO$ and $B \in \mathcal{B}$. The embedding (3.5) follows from inequality (3.6). ∎

The condition $q_{Y_\omega} < \infty$ is the best condition for the embedding (3.5) in term of Boyd's indices. For instance, when $Y_\omega = L^\infty$, the upper Boyd indices of Y_ω is infinity. We see that the embedding (3.5) does not hold because $BMO_{L^\infty} = L^\infty$ (see [7]).

Corollary 3.7. *Let $\omega \in A_\infty$. If Y_ω is an r.-i.q-B.f.s. on $(\mathbb{R}^n, \omega)$ with $q_{Y_\omega} < \infty$, then for any $f \in BMO$ and for all $\mu < \frac{K_2}{\|f\|_{BMO}}$ (K_2 is given in Proposition 3.5), we have*

$$\|e^{\mu|f-f_B|}\chi_B\|_{Y_\omega} \le C(\mu, f)\|\chi_B\|_{Y_\omega},$$

for some constant $C(\mu, f)$ independent of $B \in \mathcal{B}$.

To obtain the embedding of $BMO_{Y_\omega} \hookrightarrow BMO$, we encounter a technical obstacle. Any p-convex r.-i.q-B.f.s. Y_ω on $(\mathbb{R}^n, \omega)$ possesses two quasi-norms. The first one $\|\cdot\|_{Y_\omega}$ is not a norm but it is rearrangement-invariant. The second one $\eta(\cdot)$ is a norm but it is not necessarily rearrangement-invariant.

On one hand, the merit of having an rearrangement-invariant quasi-norm is that the Boyd type interpolation theorem can be applied to Y_ω (see [12]). On the other hand, the advantage of possessing a norm is that the associate space (Köthe dual) of Y_ω is non-trivial. Even though Y_ω does not necessarily have an rearrangement-invariant norm, the following lemma shows how to incorporate these two separated properties to obtain a generalization of Lemma 2.3 for Y_ω.

Lemma 3.8. *Let $\omega \in A_\infty$. Suppose that Y_ω is an r.-i.q-B.f.s. on $(\mathbb{R}^n, \omega)$ with $1 < p_{Y_\omega} \le q_{Y_\omega} < \infty$ and Y_ω is 1-convex. Then, the associate space of Y_ω is nontrivial and we have two constants $D_1, D_2 > 0$ such that for any Lebesgue measurable set E with $\omega(E) < \infty$,*

$$D_1\omega(E) \le \|\chi_E\|_{Y_\omega}\|\chi_E\|_{Y'_\omega} \le D_2\omega(E). \tag{3.7}$$

Proof: As Y_ω is 1-convex, it possesses an equivalent lattice norm η and, hence, $Y'_\omega \ne \{0\}$. Denote the associate norm for η by η'. Thus, $\eta'(\cdot)$ is an equivalent norm for $\|\cdot\|_{Y'_\omega}$. Furthermore, by [1], Chapter 1, Theorem 2.4, we have a constant $C > 0$ so that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\omega(x)dx \le \eta(f)\eta'(g) \le C\|f\|_{Y_\omega}\|g\|_{Y'_\omega}$$

for any $f \in Y_\omega$ and $g \in Y'_\omega$. The first inequality in (3.7) follows by taking $f = g = \chi_E$.

To establish the second inequality of (3.7), we consider the linear operator

$$P_E(f) = \left(\frac{1}{\omega(E)}\int_E f(x)\omega(x)dx\right)\chi_E$$

where E is a Lebesgue measurable set with $\omega(E) < \infty$. For any $1 \le p \le \infty$, P_E is uniformly bounded on $L^p(\omega)$. More precisely, $\|P_E\|_{L^p(\omega)\to L^p(\omega)} = 1$. Thus, for any $1 \le p, q \le \infty$ and E, P_E is of joint weak type $(p, p; q, q)$ (see [1], Chapter 4, Theorem 4.11). According to Theorem 3 of [12] or Theorem 4.4 of [8], P_E is bounded on Y_ω with operator norm independent of E. That is, there is a constant $C > 0$ such that for any Lebesgue measurable set E and any $f \in Y_\omega$,

$$\frac{1}{\omega(E)}\left|\int_E f(x)\omega(x)dx\right|\|\chi_E\|_{Y_\omega} = \|P_E(f)\|_{Y_\omega} \le C\|f\|_{Y_\omega}.$$

Then,

$$\|\chi_E\|_{Y'_\omega} = \sup\left\{\left|\int_E f(x)\omega(x)dx\right| : \|f\|_{Y_\omega} \le 1\right\} \le C\frac{\omega(E)}{\|\chi_E\|_{Y_\omega}}. \quad \blacksquare$$

We now present the main results of this paper. Theorems 3.9 and 3.10 give the characterizations of BMO by A_p weighted r.-i.q-B.f.s. for $p > 1$ and $p = 1$, respectively. To obtain Theorem 3.9, we use the openness property for A_p weight functions when $p > 1$ (see [6] Corollary 9.2.6). That is,

$$A_p = \cup_{1<r<p} A_r. \tag{3.8}$$

Theorem 3.9. *Let $1 < p < \infty$ and $\omega \in A_p$. Suppose that Y_ω is an r.-i. q-B.f.s. on $(\mathbb{R}^n, \omega)$ with $p \le p_{Y_\omega} \le q_{Y_\omega} < \infty$ and Y_ω is p-convex. Then, $Y_\omega \subseteq \mathcal{M}_0$,*

$$BMO_{Y_\omega} = BMO$$

and $\|\cdot\|_{BMO_{Y_\omega}}$ is an equivalent norm of $\|\cdot\|_{BMO}$.

Proof: We first show that $Y_\omega \subseteq \mathcal{M}_0$. Using property (3.8), we see that $\omega \in A_r$ for some r slightly smaller than p. By using Lemma 4 of [3], Y_ω is r-convex. Therefore, $\eta_{[r]}$ is well-defined. Denote the associate norm of $\eta_{[r]}$ by $\eta'_{[r]}$. From Lemma 3.8, for any $B \in \mathcal{B}$, we have $\chi_B \in Y_\omega \cap Y'_\omega$. For any $f \in Y_\omega$, the Hölder inequality ensures that

$$|f|_B = \frac{1}{|B|}\int_B |f(x)|dx \le \frac{1}{|B|}\left(\int_B |f(x)|^r \omega(x)dx\right)^{\frac{1}{r}}\left(\int_B \omega^{-\frac{r'}{r}}(x)dx\right)^{\frac{1}{r'}}$$

where r' is the conjugate of r. Using the Hölder inequality for $\eta_{[r]}$ and the definition of A_r, we obtain

$$|f|_B \le C(\eta_{[r]}(|f|^r)\eta'_{[r]}(\chi_B))^{\frac{1}{r}}\frac{1}{\omega(B)^{\frac{1}{r}}} \le C\eta(f)(\eta'_{[r]}(\chi_B))^{\frac{1}{r}}\frac{1}{\omega(B)^{\frac{1}{r}}}.$$

As Y^r_ω is $\frac{p}{r}$-convex and the lower Boyd index of Y^r_ω satisfies $p_{Y^r_\omega} = \frac{p_{Y_\omega}}{r} \ge \frac{p}{r} > 1$, we are allowed to apply Lemma 3.8 to $\|\cdot\|_{Y^r_\omega}$. Moreover, η, $\eta_{[r]}$ and $\eta'_{[r]}$ are equivalent to $\|\cdot\|_{Y_\omega}$, $\|\cdot\|_{Y^r_\omega}$ and $\|\cdot\|_{(Y^r_\omega)'}$, respectively. Thus,

$$|f|_B \le C\|f\|_{Y_\omega}\left(\frac{\omega(B)}{\|\chi_B\|_{Y^r_\omega}}\right)^{\frac{1}{r}}\frac{1}{\omega(B)^{\frac{1}{r}}} \le C\|f\|_{Y_\omega}\frac{1}{\|\chi_B\|_{Y_\omega}} < \infty.$$

That is, f_B is well-defined and $Y_\omega \subseteq \mathcal{M}_0$.

It remains to prove the embedding $BMO_{Y_\omega} \hookrightarrow BMO$. Theorem 3.3 ensures that $BMO_{L^r(\omega)} = BMO$. Thus, for any $f \in BMO_{Y_\omega}$, we obtain

$$\int_B |f(x) - f_B|^r \omega(x)dx \le \eta_{[r]}(|f-f_B|^r\chi_B)\eta'_{[r]}(\chi_B).$$

Similarly, as Y^r_ω is $\frac{p}{r}$-convex and $p_{Y^r_\omega} = \frac{p_{Y_\omega}}{r} \ge \frac{p}{r} > 1$, applying Lemma 3.8 to $\|\cdot\|_{Y^r_\omega}$ again and using the fact that $\eta_{[r]}$ and $\eta'_{[r]}$ are equivalent to $\|\cdot\|_{Y^r_\omega}$ and $\|\cdot\|_{(Y^r_\omega)'}$, respectively, we obtain

$$\begin{aligned}\int_B |f(x) - f_B|^r \omega(x)dx &\le C\frac{\big(\eta(|f-f_B|\chi_B)\big)^r\omega(B)}{\|\chi_B\|_{Y^r_\omega}}\\ &= C\frac{\||f-f_B|\chi_B\|^r_{Y_\omega}\omega(B)}{\|\chi_B\|^r_{Y_\omega}}\end{aligned}$$

where the constant $C > 0$ is independent of $B \in \mathcal{B}$ and $f \in BMO_{Y_\omega}$. Hence, the inequality

$$\frac{\|(f-f_B)\chi_B\|_{L^r(\omega)}}{\|\chi_B\|_{L^r(\omega)}} \le C\frac{\||f-f_B|\chi_B\|_{Y_\omega}}{\|\chi_B\|_{Y_\omega}}$$

is valid and the embedding $BMO_{Y_\omega} \hookrightarrow BMO$ follows apparently. ■

Using Lemma 2.3 instead of Lemma 3.8 when $\omega \in A_1$, we have the following result.

Theorem 3.10. *Let $\omega \in A_1$. If Y_ω is an r.-i.B.f.s. on $(\mathbb{R}^n, \omega)$, then*

$$BMO_{Y_\omega} = BMO$$

and $\|\cdot\|_{BMO_{Y_\omega}}$ *is an equivalent norm of* $\|\cdot\|_{BMO}$.

Acknowledge: The author would like to thank the referees for careful reading of the paper and valuable suggestions.

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