COUNTING SIMSUN PERMUTATIONS BY DESCENTS

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ABSTRACT. We count in the present work simsun permutations of length n by their number of descents. Properties studied include the recurrence relation and real-rootedness of the generating function of the number of n-simsun permutations with k descents. By means of generating function arguments, we show that the descent number is equidistributed over n-simsun permutations and n-André permutations. We also compute the mean and variance of the random variable X_n taking values the descent number of random n-simsun permutations, and deduce that the distribution of descents over random simsun permutations of length n satisfies a central and a local limit theorem as $n \to +\infty$.

1. Introduction

In a series of studies of homology representations of the symmetric group $\mathfrak{S}_n$ [15, 16], Rodica *Sim*ion and Sheila *Sun*daram introduced a special class of permutations called simsun permutations, whose definition is as follows. A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ is *simsun* if it has no double descents and with the hereditary property that when the letters $n, n-1, \ldots, 2, 1$ are erased in succession, the property of not having double descents is preserved after each erasure.

Simsun permutations are closely related to André permutations which play an important role in the theory of *cd*-indices of simplicial Eulerian posets. For results along this line, see the work of Hetyei [8], Purtill [11] and Stanley [13].

Denote by $rs(n)$ the set of simsun permutations in $\mathfrak{S}_n$. (Here, rs stands for Rodica-Sheila; see the 6th paragraph of [18] for reasons behind our choice of notation.) It is a well known classical result [1] that the nth Euler number E_n, which counts the number of alternating permutations in $\mathfrak{S}_n$, satisfies

$$\sum_{n\geqslant 0} E_n \frac{x^n}{n!} = \tan x + \sec x.$$

A remarkable property of simsun permutations is that $\# rs(n) = E_{n+1}$. Other classes of permutations counted by the Euler numbers include the André permutations and their variants, which were studied previously by Foata and Schützenberger [6].

We consider in the present work the enumeration of simsun permutations in $\mathfrak{S}_n$. In the next section, we count simsun permutations in $\mathfrak{S}_n$ by descents and obtain several properties

2000 *Mathematics Subject Classification.* Primary: 05A15; Secondary: 05A19, 05A05, 05E05, 05E10.
Key words and phrases. Simsun permutations, descents, André trees, asymptotically normal.
Work partially supported by a FRG, Hong Kong Baptist University.
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of the generating function of n-simsun permutations by descents. The sequence of coefficients of the generating function is a refinement of the classical Euler number.

In Section 3, by means of generating function arguments, we prove that the descent number is equidistributed over n-simsun permutations and n-André permutations, the latter are in turn in bijective correspondence with André trees. The descent number of André permutations corresponds to the number of leaves of André trees, thus translating the enumerative results in Section 2 into those on André trees and giving an alternative representation of simsun permutations.

In the final section, we consider the probabilistic enumeration of simsun permutation in $\mathfrak{S}_n$ and compute the mean and variance of the distribution of descents on random simsun permutations. By invoking some results of Pitman [10] and Canfield [4], we deduce central and local limit theorems for the distribution.

2. Simsun permutations

We denote, as customary, by $\mathbb{Q}$ and $\mathbb{R}$ the set of rational numbers and the set of real numbers, respectively, by $\#S$ the cardinality of a finite set S, and by $[n] := \{1, 2, \ldots, n\}$ for any positive integer n. For $n = 1, 2, 3, \ldots$, define the nth simsun polynomial $rs_n(t)$ by

$$rs_n(t) := \sum_{\sigma \in rs(n)} t^{\mathrm{des}(\sigma)},$$

where $\mathrm{des}(\sigma) := \#\{i \in [n-1] \colon \sigma_i > \sigma_{i+1}\}$ is the number of descents of σ.

The first ten members of $rs_n(t)$ are listed as follows:

$$\begin{aligned}
rs_1(t) &= 1,\\
rs_2(t) &= 1 + t,\\
rs_3(t) &= 1 + 4t,\\
rs_4(t) &= 1 + 11t + 4t^2,\\
rs_5(t) &= 1 + 26t + 34t^2,\\
rs_6(t) &= 1 + 57t + 180t^2 + 34t^3,\\
rs_7(t) &= 1 + 120t + 768t^2 + 496t^3,\\
rs_8(t) &= 1 + 247t + 2904t^2 + 4288t^3 + 496t^4,\\
rs_9(t) &= 1 + 502t + 10194t^2 + 28768t^3 + 11056t^4,\\
rs_{10}(t) &= 1 + 1013t + 34096t^2 + 166042t^3 + 141584t^4 + 11056t^5.
\end{aligned}$$

Let $rs(n,k) = \#\{\sigma \in rs(n) \colon \mathrm{des}(\sigma) = k\}$ be the number of n-simsun permutations with k descents. Then $rs_n(t)$ can be written as

$$rs_n(t) = \sum_{k \geqslant 0} rs(n,k) t^k.$$

Let $\sigma \in rs(n)$. Since σ has no double descents, any descent of σ must be followed by at least one ascent so that σ has at most $\lfloor n/2 \rfloor$ descents. It is not hard to see that the permutation

$\pi \in \mathfrak{S}_n$ defined by

$$\pi = \begin{cases} n1(n-1)2(n-2)3\cdots(\frac{n}{2}+1)(\frac{n}{2}) & \text{if } n \text{ is even,} \\ n1(n-1)2(n-2)3\cdots(\frac{n+3}{2})(\frac{n-1}{2})(\frac{n+1}{2}) & \text{if } n \text{ is odd,} \end{cases}$$

is simsun such that $\operatorname{des}(\pi) = \lfloor n/2 \rfloor$. Thus, $\deg rs_n(t) = \lfloor n/2 \rfloor$. Alternatively, this latter fact can be proved by using Theorem 1(ii) below and induction on n. Also, since the identity permutation $12\cdots n$ is the unique n-simsun permutation without descents, the constant term of $rs_n(t)$ is equal to 1, i.e., $rs_n(0) = rs(n,0) = 1$. The sequence $\{rs(n,k)\}_{k=0}^{\lfloor n/2 \rfloor}$ has already been registered by the first named author as the sequence A113897 in the Online Encyclopedia of Integer Sequences (OEIS). Define the exponential generating function of $rs_n(t)$ by

$$rs(x,t) := \sum_{n \geqslant 0} rs_n(t) \frac{x^n}{n!},$$

where $rs_0(t) := 1$ by convention.

Theorem 1. *The following hold*:

(i) $rs(n,k) = (k+1)\, rs(n-1,k) + (n-2k+1)\, rs(n-1,k-1)$;
(ii) $rs_n(t) = ((n-1)t+1)\, rs_{n-1}(t) + t(1-2t)\, rs'_{n-1}(t)$;
(iii) *for* $n > 1$, $rs_n(t)$ *is simply negatively real-rooted and interlaces* $rs_{n+1}(t)$;
(iv) *for* $n \geqslant 1$, $\{rs(n,0), rs(n,1), \ldots, rs(n, \lfloor n/2 \rfloor)\}$ *is a PF sequence*; *in particular*, *it is unimodal and log-concave*;
(v) $rs(x,t)$ *satisfies the following partial differential equation*

$$(1-xt)\frac{\partial}{\partial x} rs(x,t) + t(2t-1)\frac{\partial}{\partial t} rs(x,t) = rs(x,t)$$

in $\mathbb{Q}[t][[x]]$ *and the initial condition* $rs(0,t) = 1$;
(vi) $rs(x,t) = \left(\dfrac{\sqrt{2t-1}\sec(\frac{x}{2}\sqrt{2t-1})}{\sqrt{2t-1} - \tan(\frac{x}{2}\sqrt{2t-1})} \right)^2.$

Proof. (i) Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{n-1} \in rs(n-1)$. Denote by $D(\sigma) := \{i \in [n-2] \colon \sigma_i > \sigma_{i+1}\}$ the descent set of σ. It is a well-known fact that elements of $\mathfrak{S}_n$ can be obtained by inserting the letter n to elements of $\mathfrak{S}_{n-1}$ and the n-permutations so obtained have either the same number of, or one more, descents. Define $\sigma_{+i} := \sigma_1\cdots\sigma_i n \sigma_{i+1}\cdots\sigma_{n-1}$ for $i = 0, 1, \ldots, n-1$. If $\operatorname{des}(\sigma) = k$, then $\operatorname{des}(\sigma_{+i}) = k$ for $i \in D(\sigma) \cup \{n-1\}$; if $\operatorname{des}(\sigma) = k-1$, then $\operatorname{des}(\sigma_{+i}) = k$ for $i \in [n-1] \setminus (D(\sigma) \cup (D(\sigma)-1))$ (since simsun permutations have no double descents), where $D(\sigma) - 1 = \{i-1 \colon i \in D(\sigma)\}$. In the latter case, i can take $n-1-2(k-1)$ values. Thus,

$$rs(n,k) = (k+1)\, rs(n-1,k) + (n-2k+1)\, rs(n-1,k-1).$$

(ii) Multiplying (i) by t^k, followed by summing over $k \geqslant 1$, we have

$$\begin{aligned}\sum_{k\geqslant 1} rs(n,k)t^k &= \sum_{k\geqslant 1}(k+1)\, rs(n-1,k)t^k + \sum_{k\geqslant 1}(n-2k+1)\, rs(n-1,k-1)t^k \\ rs_n(t) - 1 &= t\sum_{k\geqslant 1} k\, rs(n-1,k)t^{k-1} + \sum_{k\geqslant 1} rs(n-1,k)t^k \\ &\quad + (n-1)t\sum_{k\geqslant 1} rs(n-1,k-1)t^{k-1} - 2t^2\sum_{k\geqslant 1}(k-1)\, rs(n-1,k-1)t^{k-2} \\ &= t\, rs'_{n-1}(t) + rs_{n-1}(t) - 1 + (n-1)t\, rs_{n-1}(t) - 2t^2\, rs'_{n-1}(t)\end{aligned}$$

which, after rearranging terms, gives (ii).

(iii) Proceed by induction on n, the case $n = 2$ being clear from the above list of simsun polynomials. Let $n \geqslant 3$ and let $t_{n-1,1} < t_{n-1,2} < \cdots < t_{n-1,\lfloor (n-1)/2\rfloor} < 0$ be the simple real zeros of $rs_{n-1}(t)$. Let also $t_{n-1,0} := -\infty$ and $t_{n-1,\lfloor (n-1)/2\rfloor+1} := 0$. By (ii), we have

$$rs_n(t_{n-1,i}) = t_{n-1,i}(1-2t_{n-1,i})\, rs'_{n-1}(t_{n-1,i})$$

for $i = 1, 2, \ldots, \lfloor (n-1)/2\rfloor$. Since $rs_{n-1}(0) = 1$ and $t_{n-1,\lfloor (n-1)/2\rfloor}$ is the first simple real zero of $rs_{n-1}(t)$ to the left of $t = 0$, $rs_{n-1}(t)$ is increasing at $t_{n-1,\lfloor (n-1)/2\rfloor}$ so that $rs'_{n-1}(t_{n-1,\lfloor (n-1)/2\rfloor})$ is positive. The simplicity of $t_{n-1,i}$'s then implies

$$\operatorname{sgn} rs'_{n-1}(t_{n-1,i}) = (-1)^{\lfloor (n-1)/2\rfloor - i}$$

so that $\operatorname{sgn} rs_n(t_{n-1,i}) = (-1)^{\lfloor (n-1)/2\rfloor+1-i}$ for $i = 1, 2, \ldots, \lfloor (n-1)/2\rfloor$. Since $\deg rs_n(t) = \lfloor n/2\rfloor = \begin{cases}\lfloor (n-1)/2\rfloor & \text{if } n \text{ is odd,}\\ \lfloor (n-1)/2\rfloor + 1 & \text{if } n \text{ is even,}\end{cases}$ we have

$$\operatorname{sgn} rs_n(t_{n-1,0}) = (-1)^{\lfloor n/2\rfloor} = \begin{cases}(-1)^{\lfloor (n-1)/2\rfloor} & \text{if } n \text{ is odd,}\\ (-1)^{\lfloor (n-1)/2\rfloor+1} & \text{if } n \text{ is even,}\end{cases}$$

Also, $\operatorname{sgn} rs_n(t_{n-1,\lfloor (n-1)/2\rfloor+1}) = 1$. Thus, if n is odd, there exist $t_{n,i} \in (t_{n-1,i}, t_{n-1,i+1})$ for which $rs_n(t_{n,i}) = 0$ for $i = 1, 2, \ldots, \lfloor (n-1)/2\rfloor = \lfloor n/2\rfloor$; if n is even, there exist $t_{n,i} \in (t_{n-1,i-1}, t_{n-1,i})$ for which $rs_n(t_{n,i}) = 0$ for $i = 1, 2, \ldots, \lfloor (n-1)/2\rfloor + 1 = \lfloor n/2\rfloor$. This completes the induction.

(iv) follows from (iii) and Theorem 2 below.

(v) Multiplying the above recurrence in (ii) by $x^{n-1}/(n-1)!$ followed by summing over $n \geqslant 2$, we get the following linear first order partial differential equation:

$$(1-xt)\frac{\partial}{\partial x}\, rs(x,t) + t(2t-1)\frac{\partial}{\partial t}\, rs(x,t) = rs(x,t).$$

It is clear that $rs(0,t) = rs_0(t) = 1$.

(vi) One can solve the above partial differential equation (PDE) by the method of characteristics [7]. Solving $dx/dt = (1-xt)/t(2t-1)$ for characteristic, we get

$$C = \frac{\sqrt{2t-1} - \tan(\frac{x}{2}\sqrt{2t-1})}{1 + \sqrt{2t-1}\tan(\frac{x}{2}\sqrt{2t-1})},$$

where C is the constant of integration. Now solving $\frac{d}{dt}\, rs = rs\,/t(2t-1)$, we get

$$rs = \frac{2t-1}{t} f(C) = \frac{2t-1}{t} f\left(\frac{\sqrt{2t-1} - \tan(\frac{x}{2}\sqrt{2t-1})}{1+\sqrt{2t-1}\tan(\frac{x}{2}\sqrt{2t-1})}\right),$$

where f is an arbitrary function. Since $rs(0,t) = 1$, we get $f(t) = (t^2+1)/2t^2$, hence the result. □

A sequence $\{a_0, a_1, \ldots, a_d\}$ of real numbers is called *log-concave* if $a_{i-1}a_{i+1} \leqslant a_i^2$ for $i = 1, 2, \ldots, d-1$. It is *unimodal* if there exists an index $0 \leqslant j \leqslant d$ such that $a_i \leqslant a_{i+1}$ for $i = 0, 1, \ldots, j-1$ and $a_i \geqslant a_{i+1}$ for $i = j, j+1, \ldots, d-1$. It is a *Pólya frequency sequence of order* r (or a PF_r *sequence*) if any minor of order r of the matrix $M = (M_{ij})_{i,j\in\mathbb{N}}$ defined by $M_{ij} = a_{j-i}$ for all $i, j \in \mathbb{N}$ (where $a_k = 0$ if $k < 0$ or $k > d$) is nonnegative. It is a *Pólya frequency sequence of infinite order* (or a *PF sequence*) if it is a PF_r sequence for all $r \geqslant 1$.

It is clear that a positive sequence $\{a_i\}$ is PF_1, and a log-concave (which is also unimodal and internal-zero free) sequence $\{a_i\}$ is PF_2.

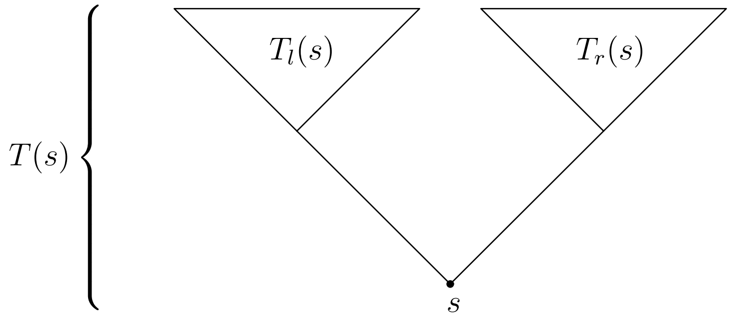
An important result concerning *PF* sequences and polynomials having only real zeros is the following [3, Theorem 2.2.4].

Theorem 2 (Aissen-Schoenberg-Whitney). *Let* $A(x) = \sum_{i=0}^{d} a_i x^i$ *be a polynomial with nonnegative coefficients. Then* $A(x)$ *has only real zeros if and only if* $\{a_0, \ldots, a_d\}$ *is a PF sequence.*

3. Connection with André permutations

We shall need in the present section the notions of minimax trees, increasing trees and André trees, where the former notion is due to Foata and Han [5]. A labelled binary tree T is said to be *minimax* if in any subtree of T, the label of the root of the subtree is either the minimum, or the maximum, of the labels of vertices of the subtree. A minimax tree T is *increasing* if the roots of all its subtrees have minimum labels. Vertices without child are called *leaves* and vertices having at least one child are called *interior* vertices. In a minimax tree T, denote by $T(s)$ the subtree of T whose root is s, and by $T_l(s)$ and $T_r(s)$ the left and right subtrees rooted at s. A minimax tree T is said to be *André* if for any interior vertex s, the right subtree $T_r(s)$ is nonempty and contains the vertex of the maximum label in $T(s)$. It follows that André trees are increasing minimax trees such that for any internal vertex s, the right subtree $T_r(s)$ is nonempty (and necessarily contain the vertex of maximum label in $T(s)$). In [5], Foata and Han call minimax trees satisfying the André property "increasing Hetyei-Reiner trees," the latter class of trees was considered in the work of Hetyei and Reiner

[9], which inspired the generalization due to Foata and Han.



It is well known that permutations can be represented as increasing binary trees [14, Chapter 1]. In [5], Foata and Han describe a bijection sending increasing binary trees of order n to permutations in $\mathfrak{S}_n$, which, when restricted to André trees, yields a bijection sending André trees of order n to "André permutations" of length n. On the other hand, according to the above definition of André trees, the rightmost leave of an André tree must have the largest label so that the corresponding "André permutation" π has its last letter the largest. Any permutation having its last letter the largest is said to be *augmented*. Thus, André trees correspond bijectively to augmented André permutations.

By restricting the bijection of Foata and Han to $rs(n)$, we get a class of increasing binary trees in bijective correspondence with $rs(n)$, which we shall describe.

THEOREM 3. *Let $\pi = \pi_1\pi_2\cdots\pi_n \in rs(n)$ and let $T(\pi)$ be its increasing binary tree representation. For any interior vertex s belonging to a left subtree, if $T_l(s)$ is non-empty, then so is $T_r(s)$.*

Proof. Let $i \in [n]$ be such that $\pi_i = s$. Since $T_l(s) \neq \varnothing$, we must have $i > 1$ and $\pi_{i-1} > \pi_i$. Since $T(\pi)$ is an increasing binary tree, if $T_r(s) = \varnothing$, then $\pi_i = s$ will be immediately preceding the smaller letter π_{i+1}, which contradicts π having no double descents. $\square$

Foata and Han [5] proved that $D_n(t) = \sum_{k\geqslant 0} d_{n,k}t^k$, the generating function of André trees of order n by the number of leaves, or the generating function of augmented André permutations (see the discussion above) of length n by the number of peaks, satisfies the following recurrence relation [5, (7.1)]:

$$D_n(t) = ntD_{n-1}(t) + t(1-2t)D'_{n-1}(t) \tag{1}$$

for $n \geqslant 1$ and $D_0(t) := 1$.

PROPOSITION 4. *For $n \geqslant 0$, we have $rs_n(t) = t^{-1}D_{n+1}(t)$, where $D_n(t) := \sum_{k\geqslant 0} d_{n,k}t^k$ is the generating function of the André trees of order n by the number of leaves.*

Proof. Let $d_n(t) := t^{-1}D_{n+1}(t)$. Substituting $D_{n+1}(t) = td_n(t)$, $D_n(t) = td_{n-1}(t)$ and $D'_n(t) = d_{n-1}(t) + td'_{n-1}(t)$ into (1) with $n+1$ in place of n, we have

$$\begin{aligned} td_n(t) &= (n+1)t^2 d_{n-1}(t) + t(1-2t)(d_{n-1}(t) + td'_{n-1}(t)) \\ &= t[((n-1)t+1)d_{n-1}(t) + t(1-2t)d'_{n-1}(t)]. \end{aligned}$$

Cancelling t from both sides, we see that $d_n(t)$ satisfies the recurrence relation of $rs_n(t)$, i.e., Theorem 1(ii). Moreover, $d_0(t) = t^{-1}D_1(t) = 1 = rs_0(t)$. Consequently, $d_n(t) \equiv rs_n(t)$, proving the proposition. □

In [5], Foata and Han show that

$$D(x,t) := \sum_{n\geqslant 0} D_n(t)\frac{x^n}{n!} = r(t)\left(\frac{1+w(t)e^{r(t)x}}{1-w(t)e^{r(t)x}}\right), \tag{2}$$

where $D_0(t) := 1$, $r(t) = (1-2t)^{1/2}$ and $w(t) = (1-r(t))/(1+r(t))$ and remark that (2) was first proved by Foata and Schützenberger [6]. In view of Proposition 4, we are able to obtain an equivalent algebraic expression for $D(x,t)$, as follows.

Since $t\, rs_n(t) = D_{n+1}(t)$ for $n \geqslant 0$, we have

$$\begin{aligned} t\sum_{n\geqslant 0} rs_n(t)\frac{x^n}{n!} &= \sum_{n\geqslant 0} D_{n+1}(t)\frac{x^n}{n!} \\ &= \frac{\partial}{\partial x}\sum_{n\geqslant 0} D_n(t)\frac{x^n}{n!} \end{aligned} \tag{3}$$

so that, by virtue of Theorem 1(iv),

$$\begin{aligned} \sum_{n\geqslant 0} D_n(t)\frac{x^n}{n!} &= \int t\sum_{n\geqslant 0} rs_n(t)\frac{x^n}{n!}\,dx \\ &= t(2t-1)\int \frac{\sec^2(\frac{x}{2}\sqrt{2t-1})}{(\sqrt{2t-1}-\tan(\frac{x}{2}\sqrt{2t-1}))^2}\,dx \\ &= \frac{2t\sqrt{2t-1}}{\sqrt{2t-1}-\tan(\frac{x}{2}\sqrt{2t-1})} + f(t) \end{aligned}$$

for some arbitrary function $f(t)$. Since $D(0,t) = D_0(t) \equiv 1$, we have that $f(t) = 1-2t$. Reporting $f(t)$ back, we finally have

$$D(x,t) = \sqrt{2t-1}\left(\frac{1+\sqrt{2t-1}\tan(\frac{x}{2}\sqrt{2t-1})}{\sqrt{2t-1}-\tan(\frac{x}{2}\sqrt{2t-1})}\right). \tag{4}$$

Alternatively, by imitating the proof of Theorem 1, one can show that $D(x,t)$ satisfies the following PDE

$$(1-xt)\frac{\partial D}{\partial x} + t(2t-1)\frac{\partial D}{\partial t} = tD$$

in $\mathbb{Q}[t][[x]]$, which, together with $D(0,t) = 1$, admits the solution given by (4).

Let $\tilde{\sigma} = \sigma_1\sigma_2\cdots\sigma_n(n+1)$ be an augmented $(n+1)$-André permutation. By restricting $\tilde{\sigma}$ to $[n]$, we obtain an André permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ such that $\mathrm{peak}(\tilde{\sigma}) = \mathrm{peak}(\sigma)$, where $\mathrm{peak}(\sigma)$ denotes the number of peaks of σ. Since σ has no double descents, every letter σ_i with $i \in D(\sigma)$ must be the last letter of an increasing run. Consequently, $\mathrm{des}(\sigma)+1 = \mathrm{peak}(\sigma)$. Since $rs_n(t) = t^{-1}D_{n+1}(t)$, it follows that the descent number des is equidistributed over n-simsun permutations and augmented $(n+1)$-André permutations. It is obvious that the map sending any augmented $(n+1)$-André permutation $\tilde{\sigma}$ to σ is a descent-preserving bijection between the set of augmented $(n+1)$-André permutations and the set of n-André permutations. Thus, des is equidistributed over n-simsun permutations and n-André permutations.

4. Central and local limit theorems

We consider in this section the probabilistic enumeration of simsun permutations by descents. Let X_n be the random variable taking values $\mathrm{des}(\sigma)$ of random simsun permutations $\sigma \in rs(n)$. We shall be interested in the distribution of X_n. We denote, as customary, by $E(X)$ and $\mathrm{Var}(X)$ the mean and variance of a random variable X.

Denote by $[t^k]f(t)$ the coefficient of t^k in the formal power series $f(t)$. Since there are $[t^k]\, rs_n(t)$ n-simsun permutations σ having $\mathrm{des}(\sigma) = k$, it follows that the probability that $P(X_n = k) = [t^k]\, rs_n(t)/\, rs_n(1)$ for $k \geqslant 0$, and that $rs_n(t)/\, rs_n(1)$ is exactly the probability generating function of X_n.

Proposition 5. *For each $n \geqslant 1$, the mean $\mu_n = E(X_n)$ and variance $\sigma_n^2 = \mathrm{Var}(X_n)$ of X_n are given by*

$$\mu_n = n + 1 - \frac{E_{n+2}}{E_{n+1}} \quad \text{and} \quad \sigma_n^2 = \frac{E_{n+3} + E_{n+2} - (n+2)E_{n+1}}{E_{n+1}} - \left(\frac{E_{n+2}}{E_{n+1}}\right)^2.$$

Proof. Setting $t = 1$ in Theorem 1(ii), we have $rs_n(1) = n\, rs_{n-1}(1) - rs'_{n-1}(1)$ so that

$$\mu_{n-1} = \frac{rs'_{n-1}(1)}{rs_{n-1}(1)} = \frac{n\, rs_{n-1}(1) - rs_n(1)}{rs_{n-1}(1)} = \frac{nE_n - E_{n+1}}{E_n} = n - \frac{E_{n+1}}{E_n},$$

since $\#\, rs(n) = E_{n+1}$. Differentiating Theorem 1(ii) once with respect to t, followed by setting $t = 1$, we have

$$rs'_n(1) = (n-1)\, rs_{n-1}(1) + n\, rs'_{n-1}(1) - 3\, rs'_{n-1}(1) - rs''_{n-1}(1)$$

so that

$$\begin{aligned} \frac{rs''_{n-1}(1)}{rs_{n-1}(1)} &= \frac{(n-1)\, rs_{n-1}(1) + (n-3)\, rs'_{n-1}(1) - rs'_n(1)}{rs_{n-1}(1)} \\ &= \frac{(n-1)E_n + (n-3)(nE_n - E_{n+1}) - ((n+1)E_{n+1} - E_{n+2})}{E_n} \\ &= \frac{E_{n+2} - (2n-2)E_{n+1} + (n^2 - 2n - 1)E_n}{E_n}. \end{aligned}$$

We thus have

$$\begin{aligned} \sigma_{n-1}^2 &= \frac{rs''_{n-1}(1)}{rs_{n-1}(1)} + \mu_{n-1} - \mu_{n-1}^2 \\ &= \frac{E_{n+2} - (2n-2)E_{n+1} + (n^2 - 2n - 1)E_n}{E_n} + \frac{nE_n - E_{n+1}}{E_n} - \left(n - \frac{E_{n+1}}{E_n}\right)^2 \\ &= \frac{E_{n+2} + E_{n+1} - (n+1)E_n}{E_n} - \left(\frac{E_{n+1}}{E_n}\right)^2, \end{aligned}$$

as desired. □

The following order notations are standard in asymptotic analysis:

$$f(n) \sim g(n) \text{ as } n \to +\infty \text{ means that } \lim_{n\to+\infty} f(n)/g(n) = 1,$$
$$f(n) = o(g(n)) \text{ as } n \to +\infty \text{ means that } \lim_{n\to+\infty} f(n)/g(n) = 0.$$

Using an asymptotic expansion of the Euler number E_n, namely,

$$E_n = 2\left(\frac{2}{\pi}\right)^{n+1} n!\left(1 + \frac{1}{(-3)^{n+1}} + \frac{1}{5^{n+1}} + \frac{1}{(-7)^{n+1}} + \frac{1}{9^{n+1}} + \cdots\right),$$

i.e., $E_n \sim 2(2/\pi)^{n+1}n!$ as $n \to +\infty$, and Proposition 5, we have the following asymptotic estimates of μ_n and σ_n^2. Note that the above asymptotic formula, with E_n denoted respectively by C_n and $(n+1)!S_{n+1}$, can be found in [12, p. 132] and [17], with the multiplicative factor 2 missing in the former reference.

Corollary 6. *We have*

$$\mu_n = \frac{(\pi-2)n-(4-\pi)}{\pi} + o(1) \quad \textit{and} \quad \sigma_n^2 = \frac{(n+2)(4+2\pi-\pi^2)}{\pi^2} + o(1),$$

as $n \to +\infty$.

For $n \in \mathbb{P}$ and $0 \leqslant k \leqslant D_n \in \mathbb{N}$, let $b(n,k) \in [0,\infty)$ and $B_n := b(n,0) + \cdots + b(n,D_n) > 0$. We say that the array $\{b(n,k)\colon n \geqslant 1,\ 0 \leqslant k \leqslant D_n\}$ satisfies a *central limit theorem with mean* μ_n *and variance* σ_n^2 provided

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\leqslant\lfloor (x)_n\rfloor} \frac{b(n,k)}{B_n} - \Phi(x) \right| = 0,$$

where $(x)_n := x\sigma_n + \mu_n$. Equivalently, we say that $\{b(n,k)\}$ is *asymptotically normal*; we also say that the array $\{b(n,k)\colon n \geqslant 1,\ 0 \leqslant k \leqslant D_n\}$ satisfies a *local limit theorem* on $S \subseteq \mathbb{R}$ if and only if

$$\lim_{n\to\infty} \sup_{x\in S} \left| \frac{b(n,\lfloor (x)_n\rfloor)}{B_n/\sigma_n} - \phi(x) \right| = 0,$$

where $\Phi(x)$ and $\phi(x)$ are, respectively, the cumulative distribution function and the probability density function of the standard normal distribution $N(0,1)$, that is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \qquad \text{and} \qquad \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

where $x \in \mathbb{R}$.

For each $n \geqslant 1$, define the standardized random variable Z_n by

$$Z_n := \frac{X_n - \mu_n}{\sigma_n}.$$

Since $\sigma_n \to +\infty$ as $n \to +\infty$ and $rs_n(t)$ is real-rooted, the following central limit theorem follows from a result of Pitman [10].

Theorem 7. *For each* $n \geqslant 1$, *the array* $\{rs(n,k)\colon 0 \leqslant k \leqslant \lfloor n/2\rfloor\}$ *satisfies a central limit theorem with mean* μ_n *and variance* σ_n^2.

By virtue of Theorem 1(iv) and a result of Canfield [4, Theorem II], the next theorem follows immediately.

Theorem 8. *For each* $n \geqslant 1$, *the array* $\{rs(n,k)\colon 0 \leqslant k \leqslant \lfloor n/2 \rfloor\}$ *satisfies a local limit theorem on* $\mathbb{R}$.

As the final remark, by virtue of the equidistribution of des over n-simsun permutations and n-André permutations, Theorem 7 and Theorem 8 also hold for André permutations.

5. Acknowledgements

The authors would like to thank the annoymous referee for corrections and suggestions which have led to considerable improvements in the accuracy and presentation of the present work.

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