ATOMIC DECOMPOSITION OF HARDY SPACES AND CHARACTERIZATION OF BMO VIA BANACH FUNCTION SPACES

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Abstract. An atomic decomposition of Hardy spaces by atoms associated with Banach function space is developed. Inspired by these decompositions, a criterion on a general Banach function space is introduced so that the characterization of BMO by using that Banach function space is valid.

1. Introduction and Background Materials

In this paper, we obtain an atomic decomposition of Hardy spaces by using atoms associated with general Banach function space. As an application of these decompositions, we generalize the characterization of BMO by using Banach function space.

We begin with some notions used in this paper. For any $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, \ r > 0\}$. For any $B \in \mathbb{B}$, denote the center and the radius of B by x_B and r_B, respectively. We write the characteristic function of B by χ_B. Let $L^1_{loc}(\mathbb{R}^n)$ denote the family of locally Lebesgue integrable functions in $\mathbb{R}^n$. Let $\mathcal{M}$ denote the set of Lebesgue measurable functions on $\mathbb{R}^n$. For any Lebesgue measurable set E, the Lebesgue measure of E is denoted by $|E|$. For any $f \in L^1_{loc}(\mathbb{R}^n)$ and $B \in \mathbb{B}$, write $f_B = \frac{1}{|B|}\int_B f(x)dx$.

We recall the definition of Banach function space (see [2, Chapter 1, Defintions 1.1 and 1.3]).

Definition 1.1. A Banach space $X \subset \mathcal{M}$ is said to be a Banach function space if it satisfies

(1) $\|f\|_X = 0 \Leftrightarrow \ f = 0$ a.e.
(2) $|g| \leq |f|$ a.e. $\Rightarrow \|g\|_X \leq \|f\|_X$
(3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$
(4) $E \in \mathcal{M}$ and $|E| < \infty \Rightarrow \chi_E \in X$
(5) $E \in \mathcal{M}$ and $|E| < \infty \Rightarrow \int_E f dx < C_E\|f\|_X, \ \forall f \in X$

for some $C_E > 0$.

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To state our results, we need to use the associate space of Banach function space (see [2, Chapter 1, Definitions 2.1 and 2.3]) and Hardy-Littlewood maximal operator (see [2, Chapter 3, Definition 3.1]).

Definition 1.2. For any Banach function space X, the associated space X' consists of all $f \in \mathcal{M}$ such that

$$\|f\|_{X'} = \sup\left\{\int_{\mathbb{R}^n} f(x)g(x)dx : \|g\|_X \le 1\right\} < \infty.$$

The associated space X' is a Banach function space, see [2, Chapter 1, Theorem 2.2 and Definition 2.3].

Definition 1.3. For any $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function $\mathrm{M}f(x)$ is defined by

$$\mathrm{M}f(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(y)|dy$$

where the supremum is taking over all $B \in \mathbb{B}$ containing x. We call the mapping M the Hardy-Littlewood maximal operator.

The main results of this paper consist of a new non-smooth atomic decomposition of Hardy spaces and a new characterization of BMO. Theorems 2.1 and 2.2 show that whenever M is bounded on X, then the non-smooth atomic decomposition of Hardy spaces generated by atoms associated with X is valid.

For the characterization of BMO, we find that if M is bounded on X', then we have the characterization of BMO by using the Banach function space X. To obtain this characterization, we use the new atomic decomposition for the Hardy space H^1 in Theorem 2.2. The proofs of these main theorems are given in Section 3.

For completeness, we review the definition of Hardy spaces. Let $\mathcal{S}$ and $\mathcal{S}'$ denote the class of tempered functions and the class of Schwartz distributions, respectively.

Definition 1.4. Let $0 < p \le 1$ and $\Phi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \Phi(x)dx \neq 0$. The Hardy space H^p consists of all $f \in \mathcal{S}'$ such that

$$\|f\|_{H^p} = \|\mathrm{M}_\Phi f\|_{L^p(\mathbb{R}^n)} < \infty$$

where

$$\mathrm{M}_\phi f(x) = \sup_{t>0} |(f * \Phi_t)(x)|$$

and $\Phi_t(x) = t^{-n}\Phi(x/t)$.

The reader is referred to [15, Chapter III] for the definition of Hardy spaces and several equivalent characterizations of Hardy spaces in term of grand-maximal function, non-tangential maximal function and atomic decompositions. In particular, the definition of H^p is independent of the function Φ used to define M_Φ whenever Φ satisfies the above conditions.

We now state the definition of the classical non-smooth atom of Hardy spaces.

Definition 1.5. Let $0 < p \leq 1 < q \leq \infty$. A family of functions $\{A_B\}_{B\in\mathbb{B}}$ is called a family of q-atoms for H^p if they satisfy

(1) $\operatorname{supp} A_B \subset 3B$;
(2) $\int_{\mathbb{R}^n} x^\gamma A(x)dx = 0, \quad \forall \gamma \in (\{0\}\cup\mathbb{N})^n$ with $|\gamma| \leq [\frac{n}{p} - n]$;
(3) $\|A_B\|_{L^q} \leq |B|^{\frac{1}{q}-\frac{1}{p}}$.

We write $\{A_B\}_{B\in\mathbb{B}} \in \mathcal{A}_{p,q}$ if $\{A_B\}_{B\in\mathbb{B}}$ is a family of q-atoms for $H^p(\mathbb{R}^n)$.

We present the classical non-smooth atomic decompositions of Hardy spaces.

Theorem 1.1. *Let $0 < p \leq 1 < q \leq \infty$. For any $f \in H^p$, there exist a family of q-atoms $\{A_B\}_{B\in\mathbb{B}}$ and $\{r_B\}_{B\in\mathbb{B}} \in l^p$ such that*

$$f = \sum_{B\in\mathbb{B}} r_B A_B$$

where the above summation converges in H^p.

Moreover, we have the following norm equivalence.

Theorem 1.2. *Let $0 < p \leq 1 < q \leq \infty$. We have*

$$\|f\|_{H^p} \approx \inf\left\{ \|\{r_B\}_{B\in\mathbb{B}}\|_{l^p} : f = \sum_{B\in\mathbb{B}} r_B A_B \textit{ and} \{A_B\}_{B\in\mathbb{B}} \in \mathcal{A}_{p,q} \right\}$$

where $f = \sum_{B\in\mathbb{B}} r_B A_B$ converges in H^p.

The reader is referred to [15, Chapter III, Section 2] for the proofs of the above results. Notice that in [15], it only establishes the above atomic decompositions in term of the ∞-atoms, the general case follows similarly, as stated in [15].

In the following section, the above notion of q-atom is extended to be a family of atoms associated with Banach function space X instead of L^q, $1 < q \leq \infty$. By using the atomic decompositions in term of this new family of atoms, we obtain the characterization of BMO by Banach function space. Recall that

$$BMO = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{BMO} = \sup_{B\in\mathbb{B}} \frac{\|(f - f_B)\chi_B\|_{L^1}}{|B|} < \infty \right\}.$$

We said that a Banach function space X can be used to give a characterization for BMO if the Banach space

$$BMO_X = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{BMO_X} = \sup_{B\in\mathbb{B}} \frac{\|(f - f_B)\chi_B\|_X}{\|\chi_B\|_X} < \infty \right\}$$

is equal to BMO and $\|\cdot\|_{BMO_X}$ and $\|\cdot\|_{BMO}$ are equivalent norms.

Theorem 2.3 shows that if the Hardy-Littlewood maximal operator is bounded on the associate space of X, then $BMO_X = BMO$. By considering the function space $X = L^\infty$, we find that our criterion is optimal in terms of the boundedness of the Hardy-Littlewood maximal operator.

When $X = L^p$, $1 \le p < \infty$, the characterization $BMO = BMO_{L^p}$ is well known. There are several new characterizations of BMO presented in [7, 9, 11]. To present the result from [7], we recall the definitions of rearrangement-invariant Banach function space and Boyd's indices (see [2, Chapter 2, Definition 4.1] and [2, Chapter 3, Definition 5.12], respectively).

For any $f \in \mathcal{M}$ that is finite a.e., the distribution function of f, $\mu_f(\lambda)$, $\lambda > 0$, is given by

$$\mu_f(\lambda) = |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|.$$

We call that f and g are equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \ge 0$.

For the properties of the distribution function of f, the reader may consult [2, Chapter 2].

Definition 1.6. We said that a Banach function space X is rearrangement-invariant (r.-i.) if for any pair of equimeasurable functions $f, g \in X$, we have $\|f\|_X = \|g\|_X$.

Boyd introduced the Boyd indices in [3]. The Boyd indices play an important role on the interpolation of linear operators on r.-i. Banach function spaces (see [2, Chapter 3]). We use the definition of Boyd's indices from [7, Defintion 1.1].

Definition 1.7. For each $t > 0$ and $f \in \mathcal{M}$, let E_t denote the dilation operator defined by

$$(E_t f)(x) = f(tx), \quad x \in \mathbb{R}^n.$$

The Boyd indices of a r.-i. Banach function space X are the numbers defined by

$$\underline{\alpha}_X = \sup_{0<t<1} \frac{\log(\|E_{1/t}\|_{X\to X})}{n \log t}, \quad \overline{\alpha}_X = \inf_{1<t<\infty} \frac{\log(\|E_{1/t}\|_{X\to X})}{n \log t}$$

where $\|E_{1/t}\|_{X\to X}$ is the operator norm of the linear operator, $E_t : X \to X$.

The reader is referred to [8, Definition 4.2] where the notion of Boyd's indices is extended to quasi-Banach function spaces and also referred to [10] where the notion of Boyd's indices is generalized to general Banach function spaces.

In [7], we find that if X is a r.-i. Banach function space with its Boyd's index $\underline{\alpha}_X$ satisfying $0 < \underline{\alpha}_X$, then X gives a characterization for BMO. In addition, the identification $BMO_X = BMO$ is proved in [7, Theorem 8].

The result in [7] is further extended the characterization of BMO to r.-i. Banach function spaces X on $(\mathbb{R}^n, \omega)$ where ω is an A_p-weighted measure in [9]. It shows that a Banach space geometric notion, p-convexity, is involved on the validity of the characterization $BMO_X = BMO$. More precisely, in [9, Theorem 3.5], we find that if an r.-i. quasi-Banach function space Y_ω on $(\mathbb{R}^n, \omega)$ satisfies $p \le p_{Y_\omega} \le q_{Y_\omega} < \infty$ and Y_ω is p-convex, then $BMO =$

BMO_{Y_ω}. Notice that the Boyd indices defined in [7] is the reciprocal of the one used in [2, 9].

In addition, we can also use the variable Lebesgue spaces to characterize BMO, see [11, Lemma 3]. For the definition of variable Lebesgue space, the reader is referred to [6, 12]. A brief introduction of variable Lebesgue space is also provided in the following section.

2. Characterization of BMO and atomic decomposition of H^p

We present our main results in this section, the proofs of our theorems are given in Section 3.

Definition 2.1. A Banach function space X belongs to the class $\mathbb{M}$ if the Hardy-Littlewood maximal operator M is bounded on X.

We write $X \in \mathbb{M}'$ if $X' \in \mathbb{M}$.

We begin with the atomic decomposition of Hardy spaces with the following family of atoms associated with X.

Definition 2.2. Let $0 < p \leq 1$ and X be a Banach function space. A family of functions $\{A_B\}_{B\in\mathbb{B}}$ belongs to $\mathcal{A}_{p,X}$ if

$$\text{supp} A_B \subset 3B, \tag{2.1}$$

$$\int_{\mathbb{R}^n} x^\gamma A_B(x)dx = 0, \quad |\gamma| \leq [\tfrac{n}{p} - n], \quad \gamma \in (\{0\} \cup \mathbb{N})^n, \tag{2.2}$$

$$\|A_B\|_X \leq |B|^{-\frac{1}{p}} \|\chi_B\|_X. \tag{2.3}$$

We call A_B non-smooth (p, X) atom.

The above family of non-smooth atoms generates a new type of atomic decomposition for Hardy spaces. It is a generalization of the well-known classical non-smooth atomic decomposition of Hardy spaces.

Theorem 2.1. *Let $0 < p \leq 1$ and X be a Banach function space. If $X \in \mathbb{M}$, then for any $f \in H^p$, there exist a sequence $\{r_B\}_{B\in\mathbb{B}} \in l^p$ and a family $\{A_B\}_{B\in\mathbb{B}} \in \mathcal{A}_{p,X}$ such that*

$$f = \sum_{B\in\mathbb{B}} r_B A_B$$

where the above sum converges in H^p.

Theorem 2.2. *Let $0 < p \leq 1$ and X be a Banach function space. If $X \in \mathbb{M}$, then*

$$\|f\|_{H^p} \approx \inf \left\{ \|\{r_B\}_{B\in\mathbb{B}}\|_{l^p} : f = \sum_{B\in\mathbb{B}} r_B A_B \text{ and } \{A_B\}_{B\in\mathbb{B}} \in \mathcal{A}_{p,X} \right\}$$

where $f = \sum_{B\in\mathbb{B}} r_B A_B$ converges in H^p.

When X is a r.-i. Banach function space, the preceding results are obtained in [7, Theorem 6].

For some criteria so that the Hardy-Littlewood maximal operator is bounded on a general Banach function space, the reader may consult [13].

Motivated by the above theorems, we find that X can be used to give a characterization of BMO provided that $X \in \mathbb{M}'$.

Theorem 2.3. *If $X \in \mathbb{M}'$, then there exist two constants $0 < A \leq B$ such that for any locally integrable function f,*

$$A\|f\|_{BMO} \leq \|f\|_{BMO_X} \leq B\|f\|_{BMO}.$$

That is, BMO is equal to BMO_X and $\|\cdot\|_{BMO_X}$ and $\|\cdot\|_{BMO}$ are equivalent norms.

Whenever X is a r.-i. Banach function space with its Boyd's indices satisfying $0 < \underline{\alpha}_X$, according to Lorentz-Shimogaki theorem, we have $X \in \mathbb{M}'$ (see [2, Chapter 3, Theorem 5.17]). Therefore, $BMO_X = BMO$ and the norms $\|\cdot\|_{BMO_X}$ and $\|\cdot\|_{BMO}$ are mutually equivalent [7, Theorem 8].

We apply Theorem 2.3 to the variable Lebesgue spaces. Recently, there are a considerable number of researches on variable Lebesgue spaces. For completeness, we recall the definition of the variable Lebesgue space from [6, 12]. For any Lebesgue measurable function $p : \mathbb{R}^n \to [1, \infty]$, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all $f \in \mathcal{M}$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\} < \infty$$

where $\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$ and

$$\rho_p(f) = \int_{\mathbb{R}^n \setminus \mathbb{R}^n_\infty} |f(x)|^{p(x)} dx + \operatorname*{ess\,sup}_{\mathbb{R}^n_\infty} |f(x)|.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}(\mathbb{R}^n)$. The reader is referred to [5, 12, 14] for some basic properties of $L^{p(\cdot)}(\mathbb{R}^n)$.

We find that the associated space of $L^{p(\cdot)}(\mathbb{R}^n)$ is given by $L^{p'(\cdot)}(\mathbb{R}^n)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ [12, Corollary 2.7], [14, Theorem 2.1]. Moreover, in view of [5, Theorem 8.1], if $p(x)$ satisfies

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\} > 1 \text{ and } p_+ = \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\} < \infty,$$

then

$$L^{p(\cdot)}(\mathbb{R}^n) \in \mathbb{M} \Leftrightarrow L^{p'(\cdot)}(\mathbb{R}^n) \in \mathbb{M} \Leftrightarrow L^{p(\cdot)}(\mathbb{R}^n) \in \mathbb{M}'.$$

Thus, the above properties and the following consequence of Jensen's inequality

$$L^{p(\cdot)}(\mathbb{R}^n) \in \mathbb{M} \Rightarrow L^{sp(\cdot)} \in \mathbb{M}, \quad 1 \leq s < \infty,$$

provide a new proof for [11, Lemma 3].

Corollary 2.4. *Let $1 \leq s < \infty$ and $p \in \mathcal{M}$ with $p_- > 1$ and $p_+ < \infty$. If $L^{p(\cdot)}(\mathbb{R}^n) \in \mathbb{M}$, then BMO is equal to $BMO_{L^{sp(\cdot)}}$.*

There is a drawback of the preceding result. It cannot be applied to p with $p_- = 1$. Especially, it does not cover the characterization of BMO by L^1. To obtain the characterization of BMO by $L^{p(\cdot)}(\mathbb{R}^n)$ with $p_- = 1$, we need the following well known result from [4, 6] which assures the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$ for the limiting case of exponent.

Theorem 2.5. *If $p(x)$ satisfies*

(1) $1 < p_- \le p_+ \le \infty$,
(2) *(The local log-Hölder condition for $1/p(x)$): There exists $C > 0$ such that for all $|x-y| \le \frac{1}{2}$*

$$\Big|\frac{1}{p(x)} - \frac{1}{p(y)}\Big| \le \frac{C}{-\log(|x-y|)}, \tag{2.4}$$

(3) *(The log-Hölder condition at ∞ for $1/p(x)$): There exist $C > 0$ and $\frac{1}{p(\infty)}$ such that for all $x \in \mathbb{R}^n$*

$$\Big|\frac{1}{p(x)} - \frac{1}{p(\infty)}\Big| \le \frac{C}{\log(e+|x|)}, \tag{2.5}$$

then $L^{p(\cdot)} \in \mathbb{M}$.

We obtain the following characterization for $L^{p(\cdot)}(\mathbb{R}^n)$ with $1 \le p_- \le p_+ < \infty$.

Theorem 2.6. *If p satisfies $1 \le p_- \le p_+ < \infty$, (2.4) and (2.5), then we have the characterization $BMO = BMO_{L^{p(\cdot)}}$.*

Proof: According to the definition of p', we have

$$\Big|\frac{1}{p(x)} - \frac{1}{p(y)}\Big| = \Big|\frac{1}{p'(x)} - \frac{1}{p'(y)}\Big|, \quad \forall x, y \in \mathbb{R}^n.$$

Thus, $p(x)$ satisfies (2.4) and (2.5) if and only if $p'(x)$ does (Take $\frac{1}{p'(\infty)} = 1 - \frac{1}{p(\infty)}$). Moreover, $1 \le p_- \le p_+ < \infty$ is equivalent with $1 < p'_- \le p'_+ \le \infty$.

We are allowed to apply Theorem 2.5 on p', we have $L^{p'(\cdot)}(\mathbb{R}^n) \in \mathbb{M}$. That is, $L^{p(\cdot)}(\mathbb{R}^n) \in \mathbb{M}'$. Thus, Theorem 2.3 guarantees the validity of the characterization $BMO = BMO_{L^{p(\cdot)}}$. ∎

Furthermore, if we apply Theorems 2.1 and 2.2 to the variable exponent Lebesgue spaces, then we obtain the non-smooth atomic decompositions of Hardy spaces associated with $\mathcal{A}_{p,L^{p(\cdot)}(\mathbb{R}^n)}$ atoms. For brevity, we leave the detail to the reader.

3. Proofs of Theorems 2.2 and 2.3

The following lemma presents the Hölder inequality on X.

Lemma 3.1. *Let X be a Banach function space on $\mathbb{R}^n$. For any $f \in X$ and $g \in X'$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le \|f\|_X \|g\|_{X'}.$$

The reader may referred to [2, Chapter 1, Theorem 2.4] for the proof of the above result.

For the variable Lebesgue spaces [12], we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

where $r_p \ge 1$ is a constant determined by $p(x)$.

We establish a supporting result in order to prove our main theorems.

Lemma 3.2. *Let X be a Banach function space. If $X \in \mathbb{M} \cup \mathbb{M}'$, then there is a constant $C \ge 1$ such that*

$$|B| \le \|\chi_B\|_X \|\chi_B\|_{X'} \le C|B|, \quad \forall B \in \mathbb{B}. \tag{3.1}$$

Proof: In view of Lorentz-Luxemburg theorem (see [2, Chapter 1, Theorem 2.7]), we have $X = X''$. Hence, it suffices to establish (3.1) with the assumption $X \in \mathbb{M}$.

The Hölder inequality on X yields the first inequality in (3.1).

For any $B \in \mathbb{B}$, we consider the projection

$$(P_B g)(y) = \left(\frac{1}{|B|} \int_B |g(x)|dx \right) \chi_B(y).$$

There exists a constant $C > 0$ such that for any $B \in \mathbb{B}$, $P_B(f) \le C\,\mathrm{M}(f)$. Hence, $\sup_B \|P_B\|_{X \to X} \le C \|\,\mathrm{M}\,\|_{X \to X}$.

Furthermore, the definition of associate space ensures that

$$\|\chi_B\|_{X'} \|\chi_B\|_X = \sup \left\{ \left| \int_B g(x)dx \right| \|\chi_B\|_X : g \in X,\ \|g\|_X \le 1 \right\} \le C|B|. \quad \blacksquare$$

We now ready to present the proofs of Theorems 2.2 and 2.3. We use the ideas from [7, Theorems 6 and 8].

Proof of Theorem 2.2:

According to the classical non-smooth atomic decomposition of H^p (see Theorems 1.1 and 1.2), for any $f \in H^p$, we have a family of scalars $r = \{r_B\}_{B \in \mathbb{B}} \in l^p$ satisfying $\|r\|_{l^p} \le C\|f\|_{H^p}$ for some constant $C > 0$ independent of f and a family of atoms $\{A_B\}_{B \in \mathbb{B}} \in \mathcal{A}_{p,\infty}$ such that $f = \sum_{B \in \mathbb{B}} r_B A_B$. The definition of associate norm, Lemma 3.1 and the definition of $\mathcal{A}_{p,\infty}$ yield

$$\|A_B\|_X \le C\|A_B\|_{L^\infty} \|\chi_{3B}\|_X \le C\|\chi_{3B}\|_X |B|^{-\frac{1}{p}}$$

for some constant C independent of f. Let $B = B(x_0, r)$. We have

$$C^{-1}4^{-n}\chi_{3B}(x) = \chi_{3B}(x) \frac{1}{C4^n r^n} \int_{|x-y| \le 4r} \chi_B(y)dy \le M(\chi_B)(x).$$

Since $X \in \mathbb{M}$, we find that $\|\chi_{3B}\|_X \le C\|\chi_B\|_X$ for some $C > 0$. Therefore, $\{CA_B(x)\}_{B \in \mathbb{B}} \in \mathcal{A}_{p,X}$ for some $C > 0$. That is, we obtain an atomic decomposition for any $f \in H^p$ with non-smooth (p, X) atoms.

It remains to show that for any $\{A_B(x)\}_{B\in\mathbb{B}} \in \mathcal{A}_{p,X}$,

$$\|A_B\|_{H^p} \leq C \tag{3.2}$$

for some constant $C > 0$ independent of $B \in \mathbb{B}$. We use the maximal function characterization of H^p to obtain our result.

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be nonnegative, radial and radially decreasing function satisfying $\operatorname{supp}\Phi \subset B(0,1)$ and $\int_{\mathbb{R}^n} \Phi \neq 0$. According to [15, Chapter 2, 2.1], we have $\mathrm{M}_\Phi(f) \leq C\,\mathrm{M}(f)$ for some constant $C > 0$ independent of f, we have

$$\|\mathrm{M}_\Phi(f)\|_X \leq C\|f\|_X \tag{3.3}$$

for some constant $C > 0$ independent of f.

Let $B = B(a,r)$ where $a \in \mathbb{R}^n$ and $r > 0$. According to the maximal function characterization of H^p, we find that

$$\begin{aligned}\|A_B\|_{H^p} = \|\mathrm{M}_\Phi(A_B)\|_{L^p} &\leq C(\|\chi_{3B}\,\mathrm{M}_\Phi(A_B)\|_{L^p} + \|(1-\chi_{3B})\,\mathrm{M}_\Phi(A_B)\|_{L^p})\\ &= I + II.\end{aligned}$$

The belonging $X \in \mathbb{M}$ and inequality (3.3) assert that

$$\begin{aligned}I &\leq \|\chi_{3B}\,\mathrm{M}_\Phi(A_B)\|_{L^1}\|\chi_{3B}\|_{L^1}^{\frac{1}{p}-1} \leq \|\mathrm{M}_\Phi(A_B)\|_X\|\chi_{3B}\|_{X'}|B|^{\frac{1}{p}-1}\\ &\leq C\|A_B\|_X\|\chi_{3B}\|_{X'}|B|^{\frac{1}{p}-1} \leq C.\end{aligned}$$

We use the vanishing moment conditions satisfied by A_B to estimate II. Write $N_p = [\frac{n}{p} - n]$. Let $N > n + N_p + 1$. In view of the fact that $x \notin 3B$ and $t > 0$, we have

$$\begin{aligned}&|(A_B * \Phi_t)(x)|\\ &= t^{-n}\left|\int_{3B} A_B(y)\big(\Phi_t(x-y) - \sum_{|\gamma|\leq N_p}\frac{y^\gamma}{\gamma!}\partial^\gamma\Phi_t(x)\big)dy\right|\end{aligned}$$

By using the differential form of the remainder term for Taylor's expansion, we obtain

$$\begin{aligned}&|(A_B * \Phi_t)(x)|\\ &\leq Ct^{-n-N_p-1}(1+t^{-1}|x-a|)^{-N}\int_{3B}|A_B(y)||y-a|^{N_p+1}dy\\ &\leq Ct^{-n-N_p-1}(1+t^{-1}|x-a|)^{-N}|B|^{\frac{N_p+1}{n}}\|A_B\|_X\|\chi_{3B}\|_{X'}\\ &\leq Ct^{-n-N_p-1}(1+t^{-1}|x-a|)^{-N}|B|^{\frac{N_p+1}{n}+1-\frac{1}{p}}\end{aligned}$$

for some $C > 0$.

Recall that $B = B(a,r)$ where $a \in \mathbb{R}^n$ and $r > 0$. For any $x \notin 3B$,

$$\mathrm{M}_\Phi(A_B)(x) \leq Cr^{N_p+1+n-\frac{n}{p}}|x-a|^{-n-N_p-1}.$$

In particular, for any $x \in B(a, 3^{j+1}r)\backslash B(a, 3^j r)$, $j \in \mathbb{N}$ with $j \geq 1$, we have

$$\int_{B(a,3^{j+1}r)\backslash B(a,3^j r)} |\,\mathrm{M}_\Phi(A_B)(x)|^p dx \leq C3^{-jp(n+N_p+1)+jn}.$$

Thus, taking summation over $j \geq 1$ on both sides of the above inequality, we find that

$$\int_{\mathbb{R}^n\backslash 3B} |\,\mathrm{M}_\Phi(A_B)(x)|^p dx \leq C\sum_{j=1}^{\infty} 3^{-jp(n+N_p+1)+jn} < C$$

because $N_p + 1 > \frac{n}{p} - n$. That is, $II \leq C$. ∎

We now give the proof of Theorem 2.3. Notice that the essence of the proof of Theorem 2.3 is the following inequality

$$\|A_B\|_{H^p} \leq C\|A_B\|_X\|\chi_{3B}\|_{X'}|B|^{\frac{1}{p}-1}.$$

This inequality is already established in the proof of Theorem 2.2.

Proof of Theorem 2.3:
Let $f \in BMO_X$. By using the Hölder inequality for X, we find that for any $B \in \mathbb{B}$,

$$\frac{1}{|B|}\int_B |f(x) - f_B|dx \leq \frac{\|(f - f_B)\chi_B\|_X\|\chi_B\|_{X'}}{|B|} \leq C\frac{\|(f - f_B)\chi_B\|_X}{\|\chi_B\|_X}.$$

For the reverse direction, we use the fact that the dual space of H^1 is BMO.

According to [2, Chapter 1, Theorem 2.9], we have a $g \in X'$ with $\|g\|_{X'} \leq 1$ and $\operatorname{supp} g \subset B$ such that

$$\|(f - f_B)\chi_B\|_X \leq 2\left|\int_B g(x)(f(x) - f_B)dx\right|.$$

For any $B \in \mathbb{B}$, we have a $\tilde{B} \in \mathbb{B}$ satisfying $|B| = |\tilde{B}|$, $B \cap \tilde{B} = \emptyset$ and $\mathrm{dist}(B, \tilde{B}) = 0$. The family of functions $\{A_B\}_{B\in\mathbb{B}}$ defined by

$$A_B(x) = \begin{cases} g(x), & x \in B \\ -\dfrac{1}{|B|}\int_B g(x)dx, & x \in \tilde{B} \\ 0, & x \in \mathbb{R}^n\backslash(B \cup \tilde{B}), \end{cases}$$

satisfies (2.1) and (2.2) with $\gamma = 0$.

In addition, inequality (3.1) assures that

$$\|A_B\|_{X'} \leq \|g\|_{X'} + \frac{1}{|B|}\|g\|_{X'}\|\chi_{3B}\|_X\|\chi_{3B}\|_{X'} \leq C. \tag{3.4}$$

Theorem 2.2 asserts that $A_B \in H^1$ with

$$\|A_B\|_{H^1} \leq C\|\chi_{3B}\|_X.$$

As $(H^1)^* = BMO$, we have

$$\frac{\|(f-f_B)\chi_B\|_X}{\|\chi_B\|_X} \leq \frac{2}{\|\chi_B\|_X}\left|\int_B g(x)(f(x)-f_B)dx\right|$$
$$= \frac{2}{\|\chi_B\|_X}\left|\int_{\mathbb{R}^n} A_B(x)(f(x)-f_B)\chi_B(x)dx\right|$$
$$\leq \frac{2\|A_B\|_{H^1}\|f\|_{BMO}}{\|\chi_{3B}\|_X} \leq C\|f\|_{BMO}. \quad \blacksquare$$

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References

[1] Aoyama, H. Lebesgue spaces with variable exponent on a probability space, *Hiroshima Math. J.* **39** (2009), 207-216.
[2] C. Bennett, and R. Sharpley, Interpolations of Operators, Academic Press, New York, 1988.
[3] Boyd, D. Indices of function spaces and their relationship to interpolation, *Canad. J. Math.* **21** (1969), 1245-1254.
[4] Cruz-Uribe, D., Diening, L., and Fiorenza, A. A new proof of the boundedness of maximal operators on variable Lebesgue spaces *Boll. Unione mat. Ital* **2(1)** (2009), 151-173.
[5] Diening, L. Maximal function on Orlicz-Musielak spaces and generalized Lebesgue space, *Bull. Sci. Math.* **129** (2005), 657-700.
[6] Diening, L., Harjulehto, P., Hästö, P., Mizuta, Y., and Shimomura, T. Maximal functions in variable exponent spaces: limiting cases of the exponent *Ann. Acad. Sci. Fenn. Math.* **34** (2009), 503-522.
[7] Ho, K.-P. Characterization of BMO in terms of rearrangement-invariant Banach function spaces *Expo. Math.* **27** (2009) 363-372.
[8] Ho, K.-P. Littlewood-Paley spaces *Math. Scand.* **108** (2011) 77-102.
[9] Ho, K.-P. Characterizations of BMO by A_p weights and P-convexity *Hiroshima Math. J.* **41** (2011) 153-165.
[10] Ho, K.-P. Generalized Boyd's indices and applications (accepted by Analysis (Munich)).
[11] Izuki, M. Boundedness of commutators on Herz spaces with variable exponent, *Rend. Circ. Mat. Palermo (2)* **59** (2010), 199-213.
[12] Kováčik O., and Rákosník, J. On spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Czechoslovak Math. J.* **41** (1991), 592-618.
[13] Lerner, A., and Ombrosi, S., A boundedness criterion for general maximal operators *Publ. Mat.*, **54** (2010), 53-71.
[14] Lukeš, J., Pick, L., and Pokorný, D. On geometric properties of the spaces $L^{p(x)}$, *Rev. Mat. Complut.* **24** (2011), 115-130.
[15] Stein, E. Harmonic Analysis, Princeton University Press 1993.

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