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Dynamics of Coherent Structures in the Coupled Complex Ginzburg-Landau Equations

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ABSTRACT

In this article, we study the modified Hirota bilinear method to construct some exact analytical solutions of the complex Ginzburg–Landau equations (CGLEs). CGLEs are intensively studied models of pattern formation in nonlinear dissipative media, with applications to biology, hydrodynamics, nonlinear optics, plasma physics, reaction–diffusion systems and many other fields. A system of two coupled CGLEs modeling the propagation of pulses under the combined influence of dispersion, self and cross phase modulations, linear and nonlinear gain and loss will be discussed. A solitary pulse (SP) is a localized wave form and a front (also termed as shock) refers to a transition connecting two constant, but unequal, asymptotic states. A SP-front pair solution can be analytically obtained by the modified Hirota bilinear method. These wave configurations are dictated by a system of six nonlinear algebraic equations, allowing the amplitudes, wave-numbers, frequency, and velocities to be determined. The final exact solution can then be computed by employing the Groebner basis method in the computer software Maple.

1. INTRODUCTION

The complex Ginzburg–Landau equations (CGLEs) govern the dynamics of patterns in nonlinear dissipative media, and arise in many disciplines, e.g. biology, chemical reactions, diffusion, hydrodynamics, optics, plasma physics and many other fields. The dynamics and propagation of the pulses are governed by the combined influence of dispersion, self and cross phase modulations, linear and nonlinear gain/loss. Many varieties of modes have been established, with the well known examples being (a) bright (or localized) solitary pulses, (b) dark pulses with minimum in intensity or holes, (c) kinks (also termed shocks or wave front solutions), transitions joining two constant, but unequal, asymptotic states. Comprehensive reviews have been given [1, 2, 3, 4].

The primary focus in the paper is a system of two waveguides governed by *two coupled* CGLEs. Conditions for the presence of a shock / wave front in one channel, and a bright solitary pulse (SP) in the other, will be elucidated. The words ‘bright SP’ / ‘dark SP’ are borrowed from optics, and refer to a ‘localized pulse’ / ‘localized minimum in a constant intensity background’ respectively. Most works in the existing literature focus either on the ‘bright – bright SPs’ situation or a ‘bright – dark SPs’ pair. Hence the present configuration of ‘bright SP – shock’ in the two waveguides would be novel. The word ‘soliton’ will occasionally be employed loosely here to substitute for solitary pulse, without implying integrability of the equations.

A brief review will provide additional motivation for the present work. CGLEs where the carrier wave packets possess a difference in group velocities can be discussed in the terminology of sources and sinks, and may help in the understanding of spatiotemporal chaos [5 – 6]. Front solutions are also termed 'domain walls' in the literature. CGLEs with spatially dependent coupling coefficients will be relevant to rotating fluid flow in narrow annulus, or large aspect ratio system with poor heat conduction coefficients [7]. In modeling convection and liquid crystals, fronts in CGLEs with resonant temporal forcing can result in 'tunable' mechanism for stabilizing these wave pulses [8].

Considerable analytical progress can be made if one of the two coupled CGLEs exhibits substantial simplifications, e.g. consisting of linear damping alone or displaying an absence of dispersion [9]. In the optical context, one such system of CGLEs models the 'nonreturn-to-zero' pulses by a superposition of two shock solutions. This dynamics is relevant to dual-core, erbium-doped, amplifier-supported fiber system. In contrast, we shall study *two nonlinearly coupled* CGLEs in this work.

Besides the search for analytical expressions for solitary waves, a crucial problem to address is the stability of the background. For an isolated CGLE, generalization of such modulation instability has been considered in recent reviews [10]. For coupled CGLEs, where one component is linear *and* dissipative, precise stability boundaries have been mapped out. For linearly coupled CGLEs with fifth order (quintic) nonlinearities, doubly asymmetric solitary pulses and breathers are possible [11].

The structure of this paper can now be explained. Solitons and fronts in isolated or uncoupled CGLE can be calculated by a modified Hirota bilinear operator (Section 2). Another critical feature is that the conventional bilinear equations must be replaced by 'trilinear equations' to compute specialized exact solutions. This is illustrated by a simple case where damping and gain are absent, i.e. CGLEs are reduced to the integrable, nonlinear Schrödinger (Manakov) equations (Section 2). The *nonlinearly* coupled Ginzburg–Landau model is then introduced (Section 3). The exact 'bright–front' pair is formulated (Section 3). Special exact solutions are then presented (Section 4), and Conclusions follow (Section 5).

2. METHOD

In the literature the method involving the use of Hirota bilinear operator has been well established in finding solitary and periodic pulses of nonlinear systems. Several modifications and improvements are at times necessary to obtain an even larger class of nonlinear waves. In the following an illustrative example will be given, namely, the modified Hirota operator by Bekki and Nozaki will first be introduced and the evolution equations recast as 'trilinear' forms will also be displayed.

The modified Hirota derivative introduced by Bekki and Nozaki earlier in [12] is

$$D_{\mu,x}^{M}(G\cdot f)=\left(\frac{\partial}{\partial x}-\mu\frac{\partial}{\partial x'}\right)^{M}G(x)\cdot f(x')\bigg|_{x=x'} . \quad (1)$$

where M is a positive integer, and μ can be complex.

The 'bright soliton – front' pair of CGLEs can be obtained by rewriting the partial differential equations as 'trilinear' forms with the Bekki-Nozaki modified Hirota operator. A concrete example is given in the following to illustrate the main idea. This is a simplified case where gain / loss are absent, i.e. CGLEs reduce to the coupled nonlinear Schrödinger equations.

The Manakov system is the special, integrable set of coupled nonlinear Schrödinger equations:

$$i\frac{\partial A}{\partial t}+\frac{\partial^2 A}{\partial x^2}\pm(AA^*+BB^*)A=0\,,\ i\frac{\partial B}{\partial t}+\frac{\partial^2 B}{\partial x^2}\pm(AA^*+BB^*)B=0\,, \quad (2)$$

and one set of exact periodic solutions in terms of Jacobi elliptic functions is known:

$$A = \sqrt{6}\, r\, k_0^2 \,\mathrm{sn}(rx)\,\mathrm{cn}(rx)\exp(-i\Omega_1 t), \quad B = \sqrt{6}\, r\, k_0 \,\mathrm{cn}(rx)\,\mathrm{dn}(rx)\exp(-i\Omega_2 t), \quad (3)$$

where Ω_1, Ω_2 are appropriate angular frequencies and k_0 is the modulus of the Jacobi elliptic functions. The long wave limits of (3) are the (double humped for waveguide A) solutions

$$A = \sqrt{6}\, r(\tanh rx \,\mathrm{sech}\, rx)\exp(ir^2 t)\,, \quad B = \sqrt{6}\, r(\mathrm{sech}^2\, rx)\exp(4ir^2 t)\,.$$

To derive (3) from (2) by the Hirota method, the trilinear formulations

$$A = \frac{G}{F}, \quad B = \frac{H}{F}, \qquad (4)$$

$$F\{(iD_t + D_x^2)G \cdot F\} + G\{GG^* + HH^* - D_x^2 F \cdot F\} = 0, \qquad (5a)$$

$$F\{(iD_t + D_x^2)H \cdot F\} + H\{GG^* + HH^* - D_x^2 F \cdot F\} = 0, \qquad (5b)$$

must be used. The bilinear decomposition, e.g. setting the second bracket to be zero in (5a, 5b), *cannot* be taken. However, for an uncoupled CGLE (p, q complex), we can still apply the trilinear form to obtain the shock/front solutions which are in agreement with formulas obtained earlier in the literature [12].

3. MODEL AND RESULTS

In this section we can present our target model, say, the nonlinearly coupled complex Ginzburg–Landau model and the major results about the model. We are going to employ the terminologies of nonlinear optics for discussion.

Slowly varying amplitudes of the electric fields A and B will typically be governed by the *nonlinearly* coupled CGLEs,

$$iA_t + p_1 A_{xx} + (q_1|A|^2 + q_2|B|^2)A = i\gamma_1 A, \qquad (6)$$

$$iB_t + p_2 B_{xx} + (q_1|B|^2 + q_2|A|^2)B = i\gamma_2 B. \qquad (7)$$

The interpretations and physical significance of the various terms can now be explained. The real parts of the coefficients p_1 and p_2 denote the group velocity dispersion, and the imaginary parts, if any, are associated with the physical effects of 'bandwidth limited amplification'. The real parts of the complex coefficients q_1 and q_2 account for the self– and cross–phase modulations respectively, while the imaginary parts measure the nonlinear gain/loss. The linear gain/loss of the optical waveguides is given by the real coefficients γ_1, γ_2.

To rewrite (6, 7) in terms of the operator (1), we take

$$A = \frac{G}{f^m}, \qquad B = \frac{\exp[i\xi x - i\Omega t]H}{f^n} \qquad (8)$$

where G and H are complex-valued functions, but f is real-valued, while m and n are complex numbers of the specific form (in which α and β are real)

$$m = 1+i\alpha, \qquad n = 1+i\beta. \qquad (9)$$

Using the modified Hirota's bilinear operator (1), the two trilinear reductions of (6, 7) are determined as follows:

$$f\{iD_{m,t} + p_1 D_{m,x}^2 - i\gamma_1)G \cdot f\}$$

$$+G\left\{q_1 GG^* + q_2 HH^* - \frac{m(m+1)p_1 D_x^2 f \cdot f}{2}\right\} = 0, \qquad (10)$$

$$f\{iD_{n,t} + p_2 D_{n,x}^2 + 2p_2 i\xi D_{n,x} + \Omega - p_2\xi^2 - i\gamma_2)H \cdot f\}$$

$$+H\left\{q_1 HH^* + q_2 GG^* - \frac{n(n+1)p_2 D_x^2 f \cdot f}{2}\right\} = 0. \qquad (11)$$

The 'D_x' (without the first subscript) refers to the ordinary Hirota derivative, or $\mu = 1$ in (1). We shall search for localized modes in *A* and shock / front in *B*. Next we assume expressions of the forms (in which *k* and ω are complex),

$$G = g\exp[kx - \omega t], \qquad (12)$$

$$H = h\exp[(k + k^*)x - (\omega + \omega^*)t], \qquad (13)$$

$$f = 1 + \exp[(k + k^*)x - (\omega + \omega^*)t], \qquad (14)$$

then by equating the proper powers of the exponentials, we finally obtain the target system of six nonlinear equations:

$$q_2 hh^* = q_1 gg^* - m(m+1)p_1(k + k^*)^2, \qquad (15)$$

$$i\omega = p_1 k^2 - i\gamma_1, \qquad (16)$$

$$p_1(m-1)(k + k^*)^2 + (p_1 - p_1^*)k^{*2} - 2i\gamma_1 + \frac{q_2 hh^*}{m} = 0, \qquad (17)$$

$$q_1 hh^* = q_2 gg^* - n(n+1)p_2(k + k^*)^2, \qquad (18)$$

$$-i(\omega + \omega^*) + p_2(k + k^*)^2 + 2p_2 i\xi(k + k^*) + \Omega - p_2\xi^2 - i\gamma_2 = 0, \qquad (19)$$

$$i(\omega+\omega^*)+p_2(k+k^*)^2(n-2)-2p_2\,i\xi(k+k^*)+\frac{q_1hh^*}{n}=0\,. \tag{20}$$

We can regard (15 – 20) as six complex algebraic equations for the unknowns ξ (real), Ω (real), α (real, or m defined by (9)), β (real, or n defined by (9)), gg^* (real), hh^* (real), k (complex), ω (complex), whereas the parameters p_1, p_2 (complex, dispersion and bandwidth limited amplification), q_1, q_2 (complex, self/cross phase modulation and nonlinear gain/loss), γ_1, γ_2 (real, linear gain/loss) are the six coefficients given by the original equations of (6, 7). In principle, g and h can be complex, but the system (6, 7) is invariant up to a complex phase factor, and thus effectively only gg^*, hh^* matter in the final solutions. Generally speaking, locating all families of solutions for (15 – 20) is a huge undertaking. Specifically if we impose special conditions on p_1, p_2, q_1, q_2, this certainly permits significant analytical progress. In terms of physical meanings we are going to investigate the solitary pulse and kink pair solution. Finding such exact solutions for a solitary pulse-kink pair will be our goal in the following.

4. DETAILS AND DISCUSSION

By separating the real and imaginary parts the six complex equations of (15 – 20) give rise to a system of 12 nonlinear real equations for the real unknowns (k_r, k_i, ω_r, ω_i, gg^*, hh^*, α, β, ξ, Ω, γ_1, γ_2). We remark that γ_1 and γ_2 are treated with purpose as unknowns for the system and we define $k := k_r + i\ k_i$, $\omega := \omega_r + i\,\omega_i$ to simplify the writing further. Not surprisingly, the above system is still too complicated and we need to do some algebraic simplifications before we plug this into the software Maple and try to find any possible exact solutions symbolically. Before we get into the details of the simplifications, we may observe that the simplest solution can easily be found by choosing that $q_2 = q_1$ and $p_2 = p_1$. From (15, 18) and the requirement that m, n be complex numbers with real part unity, the implication is $m = n$, or equivalently $\alpha = \beta$. Unfortunately, this parameter regime only gives a plane wave in x, and does not yield a spatially localized solution. In order to locate the non-degenerate case where $\alpha \neq \beta$ we thoroughly investigated the 12 real equations and eventually made the following assumptions in order to make the algebra tractable. In this section we confine our attention to

$$q_1 = -q_2 = q_r + iq_i, \qquad p_2/s = p_1 = p_r + ip_i, \qquad p_i \neq 0 \tag{21}$$

where s is real.

Equations (15) and (18) imply that

$$\alpha\beta = -2, \text{ and } s = \alpha^2/2 > 0. \tag{22}$$

This means that p_1 and p_2 must be related to each other by a real, positive multiple. By writing the real and imaginary parts of (18) explicitly, we have a homogeneous system of two unknowns $(hh^* - gg^*)$ and k_r^2. In order to have nontrivial solutions, we deduce the condition

$$3\alpha(p_r q_r + p_i q_i) - (2-\alpha^2)(p_r q_i - p_i q_r) = 0. \tag{23}$$

This condition determines the possible values of α whenever p_r, p_i, q_r, q_i are given. Note that the product of roots is –2, being consistent with (22).

Elimination of the angular frequency parameters yields the system of four real equations with six real unknowns (k_r, k_i, hh^*, ξ, α, γ_1):

$$(\alpha q_r - 2q_i)hh^* - 2(\alpha p_r - 2p_i)(4+\alpha^2)k_r^2 + 2\alpha p_i(4+\alpha^2)k_r\xi = 0\,, \tag{24}$$

$$q_i hh^* - 2p_i(k_i^2 + (3+\alpha^2)k_r^2) + 4p_r k_r k_i - 2\alpha(\alpha p_r + 2p_i)k_r\xi - 2\gamma_1 = 0\,, \tag{25}$$

$$-q_r hh^* - 2\alpha(2\alpha p_r + 3p_i)k_r^2 + 4p_i k_r k_i + 2\alpha p_i k_i^2 + 2\alpha\gamma_1 = 0\,, \tag{26}$$

$$-q_i hh^* + 2p_i(k_r^2 - (\alpha k_r - k_i)^2) + 2\alpha(2p_r - \alpha p_i)k_r^2 - 2\gamma_1 = 0\,. \tag{27}$$

We note that solving the nonlinear system of (24 – 27) by employing suitable computer software is our next primary goal. In fact we may solve this system by using the Groebner basis method in the software Maple. The software will output several sets of common zeros of Groebner basis. Each set of common zeros of the Groebner basis is equivalent to the set of common zeros of the original set of polynomials. After some simplifications, the final result is (γ_1 being arbitrary)

$$(2\lambda\alpha + 3\mu)^2 = 9\mu^2 + 8\lambda^2, \quad \mu := p_r q_r + p_i q_i, \quad \lambda := p_r q_i - p_i q_r, \tag{28}$$

$$k_r^2 = \frac{4\gamma_1 p_i d^2}{\Theta}, \quad d := \mu(4 + 2\alpha^2 + \alpha^4) + \lambda\alpha(2-\alpha^2), \tag{29}$$

$$k_i^2 = \frac{\gamma_1 \alpha^2 [2 p_i d + \alpha(4+\alpha^2)(q_r + \alpha q_i)(p_r^2 + p_i^2)]^2}{p_i \Theta}, \tag{30}$$

$$\xi^2 = \frac{4\gamma_1 (4+\alpha^2)^2 [2 p_i \mu - \alpha^2 (p_r + \alpha p_i)\lambda - \alpha q_r (p_r^2 + p_i^2)]^2}{p_i \alpha^2 \Theta}, \tag{31}$$

$$hh^* = \frac{8\gamma_1 \alpha^2 (1+\alpha^2)(4+\alpha^2)(p_r^2 + p_i^2) p_i d}{\Theta}, \tag{32}$$

$$\Theta = \Theta(\alpha; p_r, p_i, q_r, q_i) \equiv \sum_{j=0}^{10} \varphi_j (p_r, p_i, q_r, q_i) \alpha^j, \tag{33}$$

where Θ is an auxiliary polynomial. The coefficients φ_j are given in terms of lengthy expressions listed in the Appendix in [13].

Some remarks for the exact solution given by (28 – 33):

(i) p_i is not zero. In the intermediate calculations the factor p_i appears in the denominator, and thus p_i bounded away from zero becomes critical.

(ii) Each of α, k_r, k_i, ξ may assume two possible values, one positive and one negative. We will use the notations: α^+, α^-, k_r^+, k_i^-, ξ^+, ξ^- depending on whether they are positive or negative.

(iii) Recall the explicitly written six unknowns in the solution (28 – 33), the other six unknowns (Recall the explicitly written six unknowns in the solution (28 – 33), the other six unknowns (ω_r, ω_i, gg^*, β, Ω, γ_2) can then be computed accordingly. In fact we have

1. β is determined by (22),

2. γ_2 is determined by

$$q_i hh^* - \frac{1}{2} p_i (16k_r^2 + 8\alpha k_r \xi + \alpha^2 \xi^2) - \gamma_2 = 0 , \tag{34}$$

3. ω_r is determined by

$$2\alpha^2 p_i k_r^2 + 2\alpha^2 p_r k_r \xi - \frac{1}{2}\alpha^2 p_i \xi^2 - \gamma_2 - 2\omega_r = 0 , \tag{35}$$

4. ω_i is determined by

$$p_r (k_i^2 - k_r^2) + 2 p_i k_r k_i - \omega_i = 0 , \tag{36}$$

5. gg^* is determined by

$$-q_r (hh^* + gg^*) + 4((2-\alpha^2) p_r - 3\alpha p_i) k_r^2 = 0 , \tag{37}$$

6. Ω is determined by

$$2\alpha^2 p_r k_r^2 - 2\alpha^2 p_i k_r \xi - \frac{1}{2}\alpha^2 p_r \xi^2 + \Omega = 0 . \tag{38}$$

The computations show that ω_r, ω_i, γ_2 and Ω may have two different expressions. They are denoted by $\omega_r^{(1)}, \omega_r^{(2)}, \omega_i^{(1)}, \omega_i^{(2)}, \gamma_2^{(1)}, \gamma_2^{(2)}, \Omega^{(1)}, \Omega^{(2)}$.

(iv) Given the values of p_r, p_i, q_r, q_i, we find that only one member of the family of solutions $(k_r^{\pm}, k_i^{\pm}, \omega_r^{(1,2)}, \omega_i^{(1,2)}, gg^*, hh^*, \alpha^{\pm}, \beta^{\pm}, \xi^{\pm}, \Omega^{(1,2)}, \gamma_1, \gamma_2^{(1,2)})$, where γ_1 is arbitrary, will satisfy the original equations.

(v) Given the values of p_r, p_i, q_r, q_i, the positiveness of gg^*, hh^*, k_r^2, k_i^2, ξ^2 will determine the sign of α and the sign of γ_1 in the exact solution. Note that although the sign of γ_1 is restricted, this does not affect the arbitrariness of γ_1 (it is still a free parameter in the exact solution).

(vi) The exact solution can finally be deduced by verifying the family of solutions with the original equations.

As an illustrative example and with the assumptions made in (21) we have $q_2 = -q_1$, and $p_2 = sp_1 = (\alpha^2/2)\, p_1$, where

$$p_1 = -2 + i, \qquad q_1 = -1 + i. \tag{39}$$

It is shown that an exact solution with a linear gain ($\gamma_1 > 0$) can be chosen:

$$\begin{cases}
\alpha = -\dfrac{\sqrt{89}-9}{2}, \qquad \beta = \dfrac{\sqrt{89}+9}{2}, \qquad \gamma_1 > 0 \quad \text{(arbitrary)}, \\
\gamma_2 = -\dfrac{22\gamma_1(3179005\sqrt{89}-29990673)}{q}, \qquad q := 76393585\sqrt{89}-720695637, \\
k = \left(\pm 18\sqrt{\dfrac{\gamma_1(7217-765\sqrt{89})}{q}}\right) + i\left(\mp\dfrac{\sqrt{89}-9}{2}\sqrt{\dfrac{\gamma_1(7032969-745493\sqrt{89})}{q}}\right), \\
\omega = \left(\dfrac{90\gamma_1(\sqrt{89}-9)^2(83\sqrt{89}-783)}{q}\right) + i\left(\dfrac{2\gamma_1(59121955\sqrt{89}-557755407)}{q}\right), \\
gg^* = \dfrac{1080\gamma_1(191907\sqrt{89}-1810447)}{q}, \\
hh^* = \dfrac{270\gamma_1(7217-765\sqrt{89})(\sqrt{89}-9)^2}{q}, \\
\xi = \pm 4\sqrt{\dfrac{\gamma_1(3890169-412357\sqrt{89})}{(\sqrt{89}-9)^2 q}}, \\
\Omega = -\dfrac{4\gamma_1(17697425\sqrt{89}-166957173)}{q}.
\end{cases} \tag{40}$$

The above numerically represented exact solution is given by

$$\begin{cases} \alpha \approx -0.21699, \quad \beta \approx 9.21699, \quad \gamma_1 > 0 \quad \text{(arbitrary)}, \\ \gamma_2 \approx -1.66057\gamma_1, \\ k \approx \left(\pm 0.75805\sqrt{\gamma_1}\right) + i\left(\mp 0.19906\sqrt{\gamma_1}\right), \\ \omega \approx \left(0.13855\gamma_1\right) + i\left(0.8\gamma_1\right), \\ gg^* \approx 7.3440\gamma_1, \\ hh^* \approx 0.09018\gamma_1, \\ \xi \approx \pm 5.34903\sqrt{\gamma_1}, \\ \Omega \approx -0.8\gamma_1. \end{cases}$$

For $\gamma_1 < 0$, similar analysis can be performed and the corresponding analytical solutions can also be computed, but details will not be pursued here.

5. CONCLUSION

In case of the conservative wave system, i.e. in the absence of gain and loss, CGLEs reduce to the nonlinear Schrödinger equations. In order to find the exact travelling wave solutions, there exists a large variety of analytical methods and the use of symbolic algebra software is well established [14].

Upon including gain/loss, a two-waveguide system will be governed by coupled CGLEs, and one model of nonlinear coupling is investigated in this work. A combination of phase locked 'localized pulse / front' solution has been investigated and such a pair is presented here via the use of trilinear equations with the Bekki-Nozaki modified Hirota operator [12]. Sets of algebraic equations defining the amplitude, phase, wave number, and frequency of the bright (localized) soliton / kink pair are established, in conjunction with the basic properties of the nonlinear dissipative media, i.e. coefficients of the coupled CGLEs. The closed-form representations of the exact solutions, for the case where the dispersion coefficients are of same signs, are obtained analytically.

Further sets of exact solutions, for the case where the dispersion coefficients are of different signs, can also be found. One delicate, but crucially important, issue is the modulation instability of the background state, and has not been fully addressed in this work. Such instability will dictate conditions on whether the wave forms will be physically observable, and will create requirements on the range of validity of the parameters. Future works along these lines of reasoning will definitely be fruitful.

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