A Project entitled

The Game Positions of Nim with a Pass

(heaps of sizes at most four)

Submitted by

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submitted to The Hong Kong Institute of Education

for the degree of Bachelor of Education (Honours)(Primary)

in *May 2015*

Declaration

I, *Wong Oi Lin*, declare that this research report represents my own work under the supervision of *Dr. Chan Wai Hong*, and that it has not been submitted previously for examination to any tertiary institution.

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26-5-2015

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Abstract

Nim, since it came to the attention of Charles Bouton in 1890s, it has been the most classic and famous combinatorial game. Hundreds of Nim variants appeared and have been investigated extensively since a century. Nim with a pass (Nim*) is a Nim variants which challenges the power of the traditional approaches in combinatorial game theory. Because of the high complexity in 3-heap Nim*, it is believed that a complete analytical solution is difficult to give. The impact of a single pass introduced into the standard game remains an important open question and none of the methods provide solution for Nim*. Nevertheless, a partial analysis of Nim* was published a half year ago and it provides solutions for Nim* played on heaps of sizes at most three. Based on this previous research, Nim* played on heaps of sizes at most four will be analyzed in this paper and the N- and P-positions will be located.

Historical review of Nim

This game was known in Europe as early as the 16th century (Jorgensen, 2009). According to Yaglom (2001), the game is derived from a similar take away game "Jian shizi" or "picking stones game", which was played in ancient China. Later, the game was spread to America by workers who were trafficked from China. However, the origin remains uncertain (Vajda, 1992), but it has gained widespread popularity because of its simplicity. In 1890s, the game came to the attention of Charles Bouton, an associate professor of mathematics at Harvard University. Nim, is the name of the game given by him and together with a complete theory of the game, including a winning strategy (Bouton, 1902). The traditional rules of Nim are stated as follows: there are n heaps, each containing a finite number of stones. Two players’ alternate turns, each time choosing a heap and removing at least one stone in the heap. The player who cannot make a move loses the game (Low and Chan, 2015). This is a 2-player game of complete information and no chance moves are involved (Nowakowski, 1998). Each player is aware of all the details of the game position at any time, and both players always have the same options of a position. In terms of combinatorial game theory, Nim is a finite impartial combinatorial game (Berlekamp, Conway and Guy, 2001).

Definition and Concepts from Combinatorial Game Theory

Some definition and concepts from combinatorial game theory will be used. P- and N-positions partition the set of all game positions. A P-position is a position which is winning for the previous player (who just moved). An N-position is a position which is winning for Next player (who is about to make a move). All terminal positions are P-positions. From every P-position, every move leads to an N-position. From every N-position, there is a move leading to a P-position.

Winning strategy of Nim

Under the simple rules of Nim, it is easy to see how to win the game if there is only one heap: take the whole heap. It's also quite easy to win at 2-heap Nim: if the heaps have unequal number of stones, then make the heaps equal. If the heaps are already equal, then you will win the game if it is your opponent's turn to move. From then on, whatever your opponent does to one heap, do the same in the other heap. For more than two heaps, it's not easy to win. Bouton (1902) published a complete analysis and proof containing a winning strategy for Nim. His strategy was based on the binary number system. Players can quickly determine appropriate moves by a simple formula. Firstly, write the number of the stones in each heap in the binary scale of notation. Then, place these numbers horizontally so that the units are in the same vertical column. Add the numbers of the same power of 2, modulo 2. This addition without carry is called Nim-addition and the result is called the Nim-sum. If the Nim-sum equal zero, it is a P-position. If the Nim sum is greater than zero, it is an N-position. Therefore, the optimal strategy is to always be converting N-positions to P-positions on each move (Bouton, 1902; Vajda, 1992).

Variants of Nim

Variants of Nim exist since ancient times. After knowing the winning strategy of original Nim, Nim played on various configurations such as circular, triangular and rectangular could be found. Nim with modified rules is also common, for example, misere Nim requires the winner avoids taking the last object. Kayles, subtraction game, Wythoff's game are famous examples of Nim variants (Albert, Nowakowski and Wolfe, 2007). Besides, Nim with a Pass also draws people’s attention. It is denoted by Nim* and played like ordinary Nim, with the option of a single pass which can be used by exactly one player. Once the pass option is used, it cannot be used again and the game continues as ordinary Nim. The pass option can be used at any time except at the end of the game. Same as ordinary Nim, the player who cannot make a move loses the game (Low and Chan, 2015).

Researches on Nim with a Pass

When a pass is introduced to Nim, it has challenge the power of the traditional approaches in Combinatorial Game Theory. The late mathematician David Gale (1974) even offered a prize to the person who firstly gives out a solution for 3-heap Nim*. Morrison, Friedman and Landsberg (2012) stated that many of the powerful methods of combinatorial game theory are ineffective to analyze the effects of a pass in a game. It also breaks down Bouton's beautiful and elegant solution method since that a pass move renders the game non-decomposable. They used a dynamical systems approach to analysis 3-heap Nim*. The findings show the high complexity in 3-heap Nim* and believed that it is difficult to give a complete analytical solution. The impact of a single pass introduced into the standard game remains an important open question and none of the methods provide solution for Nim*. Nevertheless, a partial analysis of Nim* was published by Low and Chan (2015) and it provides solutions for Nim* played on heaps of sizes at most three. Based on this previous research, Nim* played on heaps of sizes at most four will be analyzed below.

Notation

The following notation will be used to represent a game position for Nim*. Let integers $m_i \geq 0$ for all integers $i \geq 1$. Then, $(m_1, m_2, ..., m_i)$ denotes the game position corresponding to m_i heaps of size i. For example, $(3,0,2,7)$ denotes the game position in Nim* corresponding to heaps of sizes 1,1,1,3,3,4,4,4,4,4,4 and 4.

Position with Nim-sum even even odd

Low and Chan (2015) give a partial analysis of Nim* played on heaps of sizes at most three. It can be summarized in the Table 1. Since Nim-sum plays a very important role in the analysis the original Nim, therefore, Nim-sum of the P-positions of Nim* played on heaps of sizes at most three will be shown in Table 2.

Table 1: P-positions of Nim* played on heaps of sizes at most three under the restrictions on the total number of stones.

Nim* played on heaps of sizes	Restrictions on the total no. of stones (n)	No. of Heaps of size 1 (m_1)	No. of Heaps of size 2 (m_2)	No. of Heaps of size 3 (m_3)	Position
1	≥ 3	Odd			P
1 and 2	≥ 7	Odd	Even		P
1, 2 and 3	≥ 10	Odd	Even	Even	P
		Even	Even	Even	P

Table 2: P-positions of Nim* played on heaps of sizes at most three including its Nim-sum under the restrictions on the total number of stones.

Nim* played on heaps of sizes	Restrictions on the total no. of stones (n)	No. of Heaps of size 1 (m_1)	No. of Heaps of size 2 (m_2)	No. of Heaps of size 3 (m_3)	Nim-sum	Position
1	≥ 3	Odd			Odd	P
1 and 2	≥ 7	Odd	Even		Even Odd	P
1, 2 and 3	≥ 10	Odd	Even	Even	Even Odd	P
		Even	Even	Even		P

Therefore, it is believed that the Nim-sum of P-positions of Nim* played on heaps of sizes at most four is Even Even Odd. That is, under some restrictions on the total number of stones, Nim* played on (m_1, m_2, m_3, m_4) is a P-position if and only if (m_1, m_2, m_3, m_4) is (o, e, e, e) or (e, o, o, e). However, irregularity occurs because the terminal position $(0,0,0,0)$ is a P-position which its Nim-sum is not even even odd. Even when the total number of stones is 38, irregularities still appear and it will most likely continue. By observing the irregularities, it is found that they spread with patterns and the restrictions focus on the total numbers of heaps of sizes 1 and 3.

Location of N- and P-positions

Lemma 1. For $n = m_1 + 2m_2 + 3m_3 + 4m_4$ and $n \le k$ where k is an integer, $m_1 + m_3 < 2$ and $m_2 - m_4 < 4$, if this is a P-position when (m_1, m_2, m_3, m_4) is

(i) $(0,0,0,Y)$ and $Y = 0$ or $Y = 6p + 1$ where p is a positive integer, or

(ii) $(0, X, 0, Y)$ and X is odd, $Y = X - 3$, or

(iii) $(1, X, 0, Y)$ and $Y = X + 3p - 1$ where p is an integer, or

(iv) $(0, X, 1, Y)$ and $Y = X + 3p - 2$ where p is an integer, except the two cases, (1) $X = 0$ and $Y = 6p + 1$, and (2) X is even and $Y = X - 2$,

then for $n \le k$,

(i) $(1,1,0,e)$ is a P-position if $e = 6p$, or

(ii) $(1,0,0,e)$ is a P-position if $e = 6p + 2$, or

(iii) $(0,0,1,e)$ is a P-position if $e = 6p + 4$.

proof. For $n = m_1 + 2m_2 + 3m_3 + 4m_4$ and $n \le k$, assume that $m_1 + m_3 < 2$ and $m_2 - m_4 < 4$, this is a P-position when (m_1, m_2, m_3, m_4) is

(i) $(0,0,0,Y)$ and $Y = 0$ or $Y = 6p + 1$ where p is a positive integer, or

(ii) $(0, X, 0, Y)$ and X is odd, $Y = X - 3$, or

(iii) $(1, X, 0, Y)$ and $Y = X + 3p - 1$ where p is an integer, or

(iv) $(0, X, 1, Y)$ and $Y = X + 3p - 2$ where p is an integer, except the two cases, (1) $X = 0$ and $Y = 6p + 1$, and (2) X is even and $Y = X - 2$.

By (iii), for $X=1$, $(1,1,0,Y)$ is a P-position when $Y=3p$. When p is even, Y is even. Therefore, it is true that $(1,1,0,e)$ is a P-position if $e=6p'$ where p' is an integer. By (iii), for $X=0$, $(1,0,0,Y)$ is a P-position when $Y=3p-1$. When p is odd, Y is even. Therefore, it is true that $(1,0,0,e)$ is a P-position if $e=6p'+2$ where p' is an integer. By (iv), for $X=0$, $(0,0,1,Y)$ is a P-position when $Y=3p-2$. When p is even, Y is even. Therefore, it is true that $(1,0,0,e)$ is a P-position if $e=6p'+4$ where p' is an integer.

Hence, the claim is established.

Lemma 2. For $n=m_1+2m_2+3m_3+4m_4$ and $n \le k$ where k is an integer, $m_1+m_3<2$ and $m_2-m_4<4$, if this is a P-position when (m_1,m_2,m_3,m_4) is

(i) $(0,0,0,Y)$ and $Y=0$ or $Y=6p+1$ where p is a positive integer, or

(ii) $(0,X,0,Y)$ and X is odd, $Y=X-3$, or

(iii) $(1,X,0,Y)$ and $Y=X+3p-1$ where p is an integer, or

(iv) $(0,X,1,Y)$ and $Y=X+3p-2$ where p is an integer, except the two cases, (1) $X=0$ and $Y=6p+1$, and (2) X is even and $Y=X-2$,

then for $n=k+1$, it is an N-position when (m_1,m_2,m_3,m_4) is $(1,0,1,e)$. This leaves an N-position for P2 to play on when P1 removes a heap of size 2 from $(1,e,0,e)$ where $m_2-m_4=4$.

i.e. $(1,e,0,e)\xrightarrow{-2}(1,o,0,e)$.

proof. Assume that for $n \le k$ where k is an integer, $m_1+m_3<2$ and $m_2-m_4<4$, this is a P-position when (m_1,m_2,m_3,m_4) is

(I) $(0,0,0,Y)$ and $Y=0$ or $Y=6p+1$ where p is a positive integer, or

(II) $(0,X,0,Y)$ and X is odd, $Y=X-3$, or

(III) $(1,X,0,Y)$ and $Y=X+3p-1$ where p is an integer, or

(IV) $(0,X,1,Y)$ and $Y=X+3p-2$ where p is an integer, except the two cases, (1) $X=0$ and $Y=6p+1$, and (2) X is even and $Y=X-2$.

For $n=k+1$, if (m_1,m_2,m_3,m_4) is $(1,0,1,e)$ and

(i) P1 uses the pass option, then $m'_1=m'_3=1$, $m'_2=0$ and m'_4 is even. i.e.

$(1,0,1,e)\xrightarrow{pass}(1,0,1,e)$

(ii) P1 removes a heap of size 1, then $m'_1=m'_2=0$, $m'_3=1$ and m'_4 is even. i.e.

$(1,0,1,e)\xrightarrow{-1}(0,0,1,e)$.

(iii) P1 removes one stone from a heap of size 3, then $m'_1=m'_2=1$, $m'_3=0$ and m'_4 is even. i.e. $(1,0,1,e)\xrightarrow{-1}(1,1,0,e)$.

(iv) P1 removes two stones from a heap of size 3, then $m'_1=2$, $m'_2=m'_3=0$ and m'_4 is even. i.e. $(1,0,1,e)\xrightarrow{-2}(2,0,0,e)$.

(v) P1 removes a heap of size 3, then $m'_1=1$, $m'_2=m'_3=0$ and m'_4 is even. i.e.

$(1,0,1,e)\xrightarrow{-3}(1,0,0,e)$.

(vi) P1 removes one stone from a heap of size 4, then $m'_1=1$, $m'_2=0$, $m'_3=2$ and m'_4 is odd. i.e. $(1,0,1,e)\xrightarrow{-1}(1,0,2,o)$.

(vii) P1 removes two stones from a heap of size 4, then $m'_1=m'_2=m'_3=1$ and m'_4 is odd. i.e. $(1,0,1,e)\xrightarrow{-2}(1,1,1,o)$.

(viii) P1 removes three stones from a heap of size 4, then $m'_1 = 2$, $m'_2 = 0$, $m'_3 = 1$ and m'_4 is odd. i.e. $(1,0,1,e) \xrightarrow{-3} (2,0,1,o)$.

(ix) P1 removes a heap of size 4, then $m'_1 = m'_3 = 1$, $m'_2 = 0$, and m'_4 is odd. i.e. $(1,0,1,e) \xrightarrow{-4} (1,0,1,o)$.

By Lemma 1, for $n \le k$, when m_4 is even, $(1,1,0,e)$ is a P-position if $e = 6p$, $(1,0,0,e)$ is a P-position if $e = 6p+2$ and $(0,0,1,e)$ is a P-position if $e = 6p+4$. Hence, this leaves a P-position for P2 to play on in one of the cases (ii), (iii) and (v) since $n' \le k$ and m'_4 is even. Therefore, $(1,0,1,e)$ is an N-position.

For $n = k+1$, if (m_1, m_2, m_3, m_4) is $(1,e,0,e)$ where $m_2 - m_4 = 4$, P1 removes a heap of size 2, then $n' \le k$, $m'_1 = 1$, m'_2 is odd, $m'_3 = 0$, m'_4 is even and $m'_2 - m'_4 = 3$.

i.e. $(1,e,0,e) \xrightarrow{-2} (1,o,0,e)$. For X is odd and $Y = X - 3$, by (II), $(1,o,0,e)$ is an N-position since removing a heap of size 1 will leave a P-position. Therefore, for $n = k+1$, this leaves an N-position for P2 to play on when P1 removes a heap of size 2 from $(1,e,0,e)$ where $m_2 - m_4 = 4$.

i.e. $(1,e,0,e) \xrightarrow{-2} (1,o,0,e)$.

Hence, the claim is established.

Lemma 3. For $n \le k$, if $m_1 + m_3 \ge 2$ or $m_2 - m_4 \ge 4$, this is a P-position when (m_1, m_2, m_3, m_4) is (o,e,e,e) or (e,o,o,e), and if $m_1 + m_3 < 2$ and $m_2 - m_4 < 4$, this is a P-position when (m_1, m_2, m_3, m_4) is

(i) $(0,0,0,Y)$ and $Y = 0$ or $Y = 6p+1$ where p is a positive integer, or

(ii) $(0, X, 0, Y)$ and X is odd, $Y = X - 3$, or

(iii) $(1, X, 0, Y)$ and $Y = X + 3p - 1$ where p is an integer, or

(iv) $(0, X, 1, Y)$ and $Y = X + 3p - 2$ where p is an integer, except the two cases, (1) $X = 0$ and $Y = 6p + 1$, and (2) X is even and $Y = X - 2$,

then for $n = k + 1$, $m_1 + m_3 < 2$ and $m_2 - m_4 < 4$, this is a P-position when (m_1, m_2, m_3, m_4) is

(i) $(0,0,0,Y)$ and $Y = 0$ or $Y = 6p + 1$ where p is a positive integer, or

(ii) $(0, X, 0, Y)$ and X is odd, $Y = X - 3$, or

(iii) $(1, X, 0, Y)$ and $Y = X + 3p - 1$ where p is an integer, or

(iv) $(0, X, 1, Y)$ and $Y = X + 3p - 2$ where p is an integer, except the two cases, (1) $X = 0$ and $Y = 6p + 1$, and (2) X is even and $Y = X - 2$.

proof. Assume that for $n \le k$, if $m_1 + m_3 \ge 2$ or $m_2 - m_4 \ge 4$, this is a P-position when (m_1, m_2, m_3, m_4) is (o, e, e, e) or (e, o, o, e), and if $m_1 + m_3 < 2$ and $m_2 - m_4 < 4$, this is a P-position when (m_1, m_2, m_3, m_4) is

(i) $(0,0,0,Y)$ and $Y = 0$ or $Y = 6p + 1$ where p is a positive integer, or

(ii) $(0, X, 0, Y)$ and X is odd, $Y = X - 3$, or

(iii) $(1, X, 0, Y)$ and $Y = X + 3p - 1$ where p is an integer, or

(iv) $(0, X, 1, Y)$ and $Y = X + 3p - 2$ where p is an integer, except the two cases, (1) $X = 0$ and $Y = 6p + 1$, and (2) X is even and $Y = X - 2$.

Therefore, otherwise, are N-positions. i.e. The two cases are N-positions. (1) $Y = X + 3p$ where p is an integer, except the two cases, (i) $X = Y = 0$, and (ii) X is odd and $Y = X - 3$, and (2) X is even, $Y = X - 2$.

Consider $n = k + 1$, $m_1 + m_3 < 2$ and $m_2 - m_4 < 4$. Then we analysis the game positions of the following cases, (1) $X = 0$, $Y = 6p + 1$ where p is a positive integer, (2) X is odd, $Y = X - 3$, (3) $Y = X + 3p - 1$ where p is an integer, and (4) for p is an integer, $Y = X + 3p - 2$ except the two cases, (i) $X = 0$ and $Y = 6p + 1$, and (ii) X is even and $Y = X - 2$.

For $X = 0$, $Y = 6p + 1$ where p is a positive integer and

(i) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes one stone from a heap of size 4, then $m'_1 = 0$, $m'_2 = X$, $m'_3 = 1$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-1} (0, X, 1, Y - 1)$.

(ii) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes two stones from a heap of size 4, then $m'_1 = m'_3 = 0$, $m'_2 = X + 1$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-2} (0, X + 1, 0, Y - 1)$.

(iii) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes three stones from a heap of size 4, then $m'_1 = 1$, $m'_2 = X$, $m'_3 = 0$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-3} (1, X, 0, Y - 1)$.

(iv) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes a heap of size 4, then $m'_1 = m'_3 = 0$, $m'_2 = X$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-4} (0, X, 0, Y - 1)$.

(v) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes two stones from a heap of size 4, then $m'_1 = 1$, $m'_2 = X + 1$, $m'_3 = 0$ and $m'_4 = Y - 1$. i.e. $(1, X, 0, Y) \xrightarrow{-2} (1, X + 1, 0, Y - 1)$.

(vi) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes three stones from a heap of size 4, then $m'_1 = m'_3 = 0$, $m'_2 = X$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-3} (0, X, 0, Y)$.

Since $X = 0$, $Y = 6p + 1$ where p is a positive integer, therefore these two cases are impossible in (i), (iii) and (iv), (1) $X' = Y' = 0$, and (2) X' is odd and $Y' = X' - 3$. Hence, this leaves an N-position for P2 in (i), (iii) and (iv) since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p'$. This leaves an N-position for P2 in (ii) since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 2$. Therefore, $(0, X, 0, Y)$ is a P-position. This leaves a P-position for P2 to play on in (v) since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 1$. Therefore, $(1, X, 0, Y)$ is an N-position. In (vi), this leaves a P-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 < 4$, $X' = 0$ and $Y' = 6p' + 1$. Hence, $(0, X, 1, Y)$ is an N-position.

For X is odd, $Y = X - 3$ and

(i) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes one stone from a heap of size 2, then $m'_1 = 1$, $m'_2 = X - 1$, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(0, X, 0, Y) \xrightarrow{-1} (1, X - 1, 0, Y)$.

(ii) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes a heap of size 2, then $m'_1 = m'_3 = 0$, $m'_2 = X - 1$ and $m'_4 = Y$. i.e. $(0, X, 0, Y) \xrightarrow{-2} (0, X - 1, 0, Y)$.

(iii) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes one stone from a heap of size 4, then $m'_1 = 0$, $m'_2 = X$, $m'_3 = 1$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-1} (0, X, 1, Y - 1)$.

(iv) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes two stones from a heap of size 4, then $m'_1 = m'_3 = 0$, $m'_2 = X + 1$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-2} (0, X + 1, 0, Y - 1)$.

(v) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes three stones from a heap of size 4, then $m'_1 = 1$, $m'_2 = X$, $m'_3 = 0$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-3} (1, X, 0, Y - 1)$.

(vi) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes a heap of size 4, then $m'_1 = m'_3 = 0$, $m'_2 = X$ and $m'_4 = Y - 1$. i.e. $(0, X, 0, Y) \xrightarrow{-4} (0, X, 0, Y - 1)$.

(vii) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes a heap of size 1, then $m'_1 = m'_3 = 0$, $m'_2 = X$ and $m'_4 = Y$. i.e. $(1, X, 0, Y) \xrightarrow{-1} (0, X, 0, Y)$.

(viii) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes a heap of size 3, then $m'_1 = m'_3 = 0$, $m'_2 = X$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-3} (0, X, 0, Y)$.

This leaves an N-position for P2 to play on in (i) and (ii) since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 2$. In (iii), (v) and (vi), this leaves an N-position for P2 since $n' \le k$, $m'_2 - m'_4 \ge 4$, (m'_1, m'_2, m'_3, m'_4) is not (o, e, e, e) or (e, o, o, e). In (iv), this leaves an N-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 \ge 4$, (m'_1, m'_2, m'_3, m'_4) is not (o, e, e, e) or (e, o, o, e). Therefore, $(0, X, 0, Y)$ is a P-position. This leaves a P-position for P2 to play on in (vii) and (viii) since $n' \le k$, $m'_2 - m'_4 < 4$, X' is odd and $Y' = X' - 3$. Hence, $(1, X, 0, Y)$ and $(0, X, 1, Y)$ are N-positions.

For $Y = X + 3p - 1$ where p is an integer and

(i) If $m_1 = m_3 = 0$, $m_2 = X$, $m_4 = Y$, X is not even and $Y \ne X - 1$, P1 removes one stone from a heap of size 4, then $m'_1 = 0$, $m'_2 = X$, $m'_3 = 1$ and $m'_4 = Y - 1$.

i.e. $(0, X, 0, Y) \xrightarrow{-1} (0, X, 1, Y - 1)$.

(ii) If $m_1 = m_3 = 0$, $m_2 = X$, $m_4 = Y$, X is even and $Y = X - 1$, P1 removes two stone from a heap of size 4, then $m'_1 = 0$, $m'_2 = X + 1$, $m'_3 = 1$ and $m'_4 = Y - 1$.

i.e. $(0, X, 0, Y) \xrightarrow{-2} (0, X + 1, 0, Y - 1)$.

(iii) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes a heap of size 1, then m'_1=0, m'_2=X, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(1, X, 0, Y) \xrightarrow{-1} (0, X, 0, Y)$.

(iv) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes one stone from a heap of size 2, then $m'_1 = 2$, $m'_2 = X - 1$, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(1, X, 0, Y) \xrightarrow{-1} (2, X - 1, 0, Y)$.

(v) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes a heap of size 2, then $m'_1 = 1$, $m'_2 = X - 1$, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(1, X, 0, Y) \xrightarrow{-2} (1, X - 1, 0, Y)$.

(vi) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes one stone from a heap of size 4, then $m'_1 = m'_3 = 1$, $m'_2 = X$ and $m'_4 = Y - 1$. i.e. $(1, X, 0, Y) \xrightarrow{-1} (1, X, 1, Y - 1)$.

(vii) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes two stones from a heap of size 4, then $m'_1 = 1$, $m'_2 = X + 1$, $m'_3 = 0$ and $m'_4 = Y - 1$. i.e. $(1, X, 0, Y) \xrightarrow{-2} (1, X + 1, 0, Y - 1)$.

(viii) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes three stones from a heap of size 4, then $m'_1 = 2$, $m'_2 = X$, $m'_3 = 0$ and $m'_4 = Y - 1$. i.e. $(1, X, 0, Y) \xrightarrow{-3} (2, X, 0, Y - 1)$.

(ix) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes a heap of size 4, then $m'_1 = 1$, $m'_2 = X$, $m'_3 = 0$ and $m'_4 = Y - 1$. i.e. $(1, X, 0, Y) \xrightarrow{-4} (1, X, 0, Y - 1)$.

(x) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, X is not even and $Y \neq X - 1$, P1 removes a heap of size 4, then $m'_1 = 0$, $m'_2 = X$, $m'_3 = 1$ and $m'_4 = Y - 1$. i.e. $(0, X, 1, Y) \xrightarrow{-4} (0, X, 1, Y - 1)$.

(xi) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$, $m_4 = Y$, X is even and $Y = X - 1$, P1 removes two stones from a heap of size 3, then $m'_1 = 1$, $m'_2 = X$, $m'_3 = 0$ and $m'_4 = Y$.

i.e. $(0, X, 1, Y) \xrightarrow{-2} (1, X, 0, Y)$.

Since $Y = X + 3p - 1$, these two cases are impossible in (i) and (x), (1) $X' = 0$ and $Y' = 6p' + 1$, and (2) X' is even and $Y' = X' - 2$. Therefore, this leaves a P-position for P2 to play on in (i) and

(x) since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 2$. In (ii), this leaves a P-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 < 4$, X' is odd and $Y' = X' - 3$. In (xi), this leaves a P-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 1$. Therefore, $(0, X, 0, Y)$ and $(0, X, 1, Y)$ are N-positions. In (iv), (vi) and (viii), this leaves an N-position for P2 to play on since $n' \le k$, $m'_1 + m'_3 \ge 2$, (m'_1, m'_2, m'_3, m'_4) is not (e, o, o, e) or (o, e, e, e). In (iii), this leaves an N-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 1$. In (ix), this leaves an N-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 2$. Since $Y = X + 3p - 1$, these two cases are impossible in (v) and (vii), (1) $X' = Y' = 0$, and (2) X' is odd and $Y' = X' - 3$. Therefore, this leaves an N-position for P2 to play on in (v) and (vii) since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p'$. Hence, $(1, X, 0, Y)$ is a P-position.

For $Y = X + 3p - 2$ where p is an integer, except the two cases, (1) $X = 0$ and $Y = 6p + 1$, and (2) X is even and $Y = X - 2$,

(i) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 removes one stone from a heap of size 2, then $m'_1 = 1$, $m'_2 = X - 1$, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(0, X, 0, Y) \xrightarrow{-1} (1, X - 1, 0, Y)$.

(ii) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes a heap of size 2, then $m'_1 = 1$, $m'_2 = X - 1$, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(1, X, 0, Y) \xrightarrow{-2} (1, X - 1, 0, Y)$.

(iii) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes one stone from a heap of size 2, then $m'_1 = m'_3 = 1$, $m'_2 = X - 1$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-1} (1, X - 1, 1, Y)$.

(iv) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes a heap of size 2, then $m'_1 = 0$, $m'_2 = X - 1$, $m'_3 = 1$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-2} (0, X - 1, 1, Y)$.

(v) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes one stone from a heap of size 3, then $m'_1 = m'_3 = 0$, $m'_2 = X + 1$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-1} (0, X+1, 0, Y)$.

(vi) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes two stones from a heap of size 3, then $m'_1 = 1$, $m'_2 = X$, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-2} (1, X, 0, Y)$.

(vii) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes a heap of size 3, then $m'_1 = m'_3 = 0$, $m'_2 = X$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-3} (0, X, 0, Y)$.

(viii) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes one stone from a heap of size 4, then $m'_1 = 0$, $m'_2 = X$, $m'_3 = 2$ and $m'_4 = Y - 1$. i.e. $(0, X, 1, Y) \xrightarrow{-1} (0, X, 2, Y-1)$.

(ix) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes two stones from a heap of size 4, then $m'_1 = 0$, $m'_2 = X + 1$, $m'_3 = 1$ and $m'_4 = Y - 1$. i.e. $(0, X, 1, Y) \xrightarrow{-2} (0, X+1, 1, Y-1)$.

(x) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes three stones from a heap of size 4, then $m'_1 = m'_3 = 1$, $m'_2 = X$ and $m'_4 = Y - 1$. i.e. $(0, X, 1, Y) \xrightarrow{-3} (1, X, 1, Y-1)$.

(xi) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes a heap of size 4, then $m'_1 = 0$, $m'_2 = X$, $m'_3 = 1$ and $m'_4 = Y - 1$. i.e. $(0, X, 1, Y) \xrightarrow{-4} (0, X, 1, Y-1)$.

This leaves a P-position for P2 to play on in (i) and (ii) since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 1$. Hence, $(0, X, 0, Y)$ and $(1, X, 0, Y)$ are N-positions. In (iii), (viii) and (x), this leaves an N-position for P2 to play on in since $n' \le k$, $m'_1 + m'_3 \ge 2$ and (m'_1, m'_2, m'_3, m'_4) is not (e, o, o, e) or (o, e, e, e). In (iv) and (ix), this leaves an N-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 1$. In (vi) and (vii), this leaves an N-position for P2 to play on since $n' \le k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 2$. Since $Y = X + 3p - 2$ where p is an integer, except the two cases, (1) $X = 0$ and $Y = 6p + 1$, and (2) X is even and $Y = X - 2$, therefore these two cases are impossible in (v) and (xi), (1) $X'=Y'=0$, and (2) X' is odd and $Y'=X'-3$. Hence, in (v) and (xi), this leaves an N-position for P2 to play on since $n'\le k$, $m'_2-m'_4<4$ and $Y'=X'+3p'$. In (ix), if $X-Y=2$, X and Y are odd, then $m'_2-m'_4=4$, X and Y are even, this leaves an N-position for P2 to play on since $n'\le k$, $m'_2-m'_4\ge 4$, (m'_1,m'_2,m'_3,m'_4) is not (e,o,o,e) or (o,e,e,e); if $X-Y\ne 2$, X and Y are not both odd, this leaves an N-position for P2 to play on since $n'\le k$, $m'_2-m'_4<4$ and $Y'=X'+3p'-1$. Hence, $(0,X,1,Y)$ is a P-position.

Finally, we show that the remaining cases are N-positions. There are three cases, (1) $Y=X+3p$ where p is an integer, except the two cases, (i) $X=Y=0$, and (ii) X is odd and $Y=X-3$, and (2) X is even, $Y=X-2$.

For $Y=X+3p$ where p is an integer, except the two cases, (i) $X=Y=0$, and (ii) X is odd and $Y=X-3$,

(i) If $m_1=m_3=0$, $m_2=X$ and $m_4=Y$, P1 removes three stones from a heap of size 4, then $m'_1=1$, $m'_2=X$, $m'_3=0$ and $m'_4=Y-1$. i.e. $(0,X,0,Y)\xrightarrow{-3}(1,X,0,Y-1)$.

(ii) If $m_1=1$, $m_2=X$, $m_3=0$ and $m_4=Y$, P1 removes a heap of size 4, then $m'_1=1$, $m'_2=X$, $m'_3=0$ and $m'_4=Y-1$. i.e. $(1,X,0,Y)\xrightarrow{-4}(1,X,0,Y-1)$.

(iii) If $m_1=0$, $m_2=X$, $m_3=1$ and $m_4=Y$, P1 removes a heap of size 2, then $m'_1=0$, $m'_2=X-1$, $m'_3=1$ and $m'_4=Y$. i.e. $(0,X,1,Y)\xrightarrow{-2}(0,X-1,1,Y)$.

In cases (i) and (ii), if $X-Y=3$, X is even and Y is odd, this leaves a P-position for P2 since $n'\le k$, $m'_2-m'_4\ge 4$ and (m'_1,m'_2,m'_3,m'_4) is (o,e,e,e); if $X-Y\ne 3$, X is not even and Y is not odd, this leaves a P-position for P2 to play on since $n'\le k$, $m'_2-m'_4<4$ and $Y'=X'+3p'-1$.

In (iii), since we do not consider X is odd and $Y = X - 3$, therefore, X' is even and $Y' = X' - 2$ is impossible. And since $Y = X + 3p$, therefore, $X' = 0$ and $Y' = 6p' + 1$ is also impossible. Then this leaves a P-position for P2 to play on since $n' \leq k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 2$. Hence, $(0, X, 0, Y)$, $(1, X, 0, Y)$ and $(0, X, 1, Y)$ are N-positions.

For X is even, $Y = X - 2$ and

(i) If $m_1 = m_3 = 0$, $m_2 = X$ and $m_4 = Y$, P1 uses the pass option, then $m'_1 = m'_3 = 0$, $m'_2 = X$ and $m'_4 = Y$. i.e. $(0, X, 0, Y) \xrightarrow{pass} (0, X, 0, Y)$.

(ii) If $m_1 = 1$, $m_2 = X$, $m_3 = 0$ and $m_4 = Y$, P1 removes a heap of size 2, then $m'_1 = 1$, $m'_2 = X - 1$, $m'_3 = 0$ and $m'_4 = Y$. i.e. $(1, X, 0, Y) \xrightarrow{-2} (1, X - 1, 0, Y)$.

(iii) If $m_1 = 0$, $m_2 = X$, $m_3 = 1$ and $m_4 = Y$, P1 removes one stone from a heap of size 3, then $m'_1 = m'_3 = 0$, $m'_2 = X + 1$ and $m'_4 = Y$. i.e. $(0, X, 1, Y) \xrightarrow{-1} (0, X + 1, 0, Y)$.

This leaves a P-position in Nim for P2 in (i). In (ii), this leaves a P-position for P2 to play on since $n' \leq k$, $m'_2 - m'_4 < 4$ and $Y' = X' + 3p' - 1$. In (iii), this leaves a P-position for P2 to play on since $n' \leq k$, $m'_2 - m'_4 < 4$, X' is odd and $Y' = X' - 3$. Hence, $(0, X, 0, Y)$, $(1, X, 0, Y)$ and $(0, X, 1, Y)$ are N-positions.

Thus, the claim is established.

Theorem 1. Suppose that Nim* is played on (m_1, m_2, m_3, m_4), if $m_1 + m_3 \geq 2$ or $m_2 - m_4 \geq 4$, this is a P-position if and only if (m_1, m_2, m_3, m_4) is (o, e, e, e) or (e, o, o, e). If $m_1 + m_3 < 2$ and $m_2 - m_4 < 4$, this is a P-position when (m_1, m_2, m_3, m_4) is

(i) $(0,0,0,Y)$ and $Y=0$ or $Y=6p+1$ where p is a positive integer, or

(ii) $(0,X,0,Y)$ and X is odd, $Y=X-3$, or

(iii) $(1,X,0,Y)$ and $Y=X+3p-1$ where p is an integer, or

(iv) $(0,X,1,Y)$ and $Y=X+3p-2$ where p is an integer, except the two cases, (1) $X=0$ and $Y=6p+1$, and (2) X is even and $Y=X-2$.

proof. Prove by induction on the total number of stones, $n=m_1+2m_2+3m_3+4m_4$.

All the possibilities for $1\le n\le 19$ are listed in Table 3 (See Appendix I).

Therefore, from the table, the claim holds for $1\le n\le 19$.

Then, by Lemma 3, for $n=20$, $m_1+m_3<2$ and $m_2-m_4<4$, this is a P-position when (m_1,m_2,m_3,m_4) is

(I) $(0,0,0,Y)$ and $Y=0$ or $Y=6p+1$ where p is a positive integer, or

(II) $(0,X,0,Y)$ and X is odd, $Y=X-3$, or

(III) $(1,X,0,Y)$ and $Y=X+3p-1$ where p is an integer, or

(IV) $(0,X,1,Y)$ and $Y=X+3p-2$ where p is an integer, except the two cases, (1) $X=0$ and $Y=6p+1$, and (2) X is even and $Y=X-2$.

Consider $n=20$ and $m_1+m_3\ge 2$.

If (m_1,m_2,m_3,m_4) is (o,e,e,e) and $m_1+m_3\ge 3$ and

(i) P1 uses the pass option, then m'_1 is odd, m'_2, m'_3 and m'_4 are even.

i.e. $(o,e,e,e)\xrightarrow{pass}(o,e,e,e)$.

(ii) P1 removes a heap of size 1, then m'_1, m'_2, m'_3 and m'_4 are even.

i.e. $(o,e,e,e)\xrightarrow{-1}(e,e,e,e)$.

(iii) P1 removes one stone from a heap of size 2, then m'_1, m'_3 and m'_4 are even, m'_2 is odd.

i.e. $(o,e,e,e)\xrightarrow{-1}(e,o,e,e)$.

(iv) P1 removes a heap of size 2, then m'_1 and m'_2 are odd, m'_3 and m'_4 are even.

i.e. $(o,e,e,e)\xrightarrow{-2}(o,o,e,e)$.

(v) P1 removes one stone from a heap of size 3, then m'_1, m'_2 and m'_3 are odd, m'_4 is even.

i.e. $(o,e,e,e)\xrightarrow{-1}(o,o,o,e)$.

(vi) P1 removes two stones from a heap of size 3, then m'_1, m'_2 and m'_4 are even, m'_3 is odd.

i.e. $(o,e,e,e)\xrightarrow{-2}(e,e,o,e)$.

(vii) P1 removes a heap of size 3, then m'_1 and m'_3 are odd, m'_2 and m'_4 are even.

i.e. $(o,e,e,e)\xrightarrow{-3}(o,e,o,e)$.

(viii) P1 removes one stone from a heap of size 4, then m'_1, m'_3 and m'_4 are odd, m'_2 is even.

i.e. $(o,e,e,e)\xrightarrow{-1}(o,e,o,o)$.

(ix) P1 removes two stones from a heap of size 4, then m'_1, m'_2 and m'_4 are odd, m'_3 is even.

i.e. $(o,e,e,e)\xrightarrow{-2}(o,o,e,o)$.

(x) P1 removes three stones from a heap of size 4, then m'_1, m'_2 and m'_3 are even, m'_4 is odd.

i.e. $(o,e,e,e)\xrightarrow{-3}(e,e,e,o)$.

(xi) P1 removes a heap of size 4, then m'_1 and m'_4 are odd, m'_2 and m'_3 are even.

i.e. $(o,e,e,e)\xrightarrow{-4}(o,e,e,o)$.

This leaves an N-position of Nim for P2 in (i) while in all other cases, since $n-4 \le n' < n$ and $m'_1 + m'_3 \ge 2$, (m'_1, m'_2, m'_3, m'_4) is not (e,o,o,e) or (o,e,e,e), this leaves an N-position for P2 to play on. Hence (o,e,e,e) is a P-position.

If (m_1, m_2, m_3, m_4) is (e,o,o,e) and $m_1 + m_3 \ge 3$ and

(i) P1 uses the pass option, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd.

i.e. $(e,o,o,e) \xrightarrow{pass} (e,o,o,e)$.

(ii) P1 removes a heap of size 1, then m'_1, m'_2 and m'_3 are odd, m'_4 is even.

i.e. $(e,o,o,e) \xrightarrow{-1} (o,o,o,e)$.

(iii) P1 removes one stone from a heap of size 2, then m'_1 and m'_3 are odd, m'_2 and m'_4 are even. i.e. $(e,o,o,e) \xrightarrow{-1} (o,e,o,e)$.

(iv) P1 removes a heap of size 2, then m'_1, m'_2 and m'_4 are even, m'_3 is odd.

i.e. $(e,o,o,e) \xrightarrow{-2} (e,e,o,e)$.

(v) P1 removes one stone from a heap of size 3, then m'_1, m'_2, m'_3 and m'_4 are even.

i.e. $(e,o,o,e) \xrightarrow{-1} (e,e,e,e)$.

(vi) P1 removes two stones from a heap of size 3, then m'_1 and m'_2 are odd, m'_3 and m'_4 are even. i.e. $(e,o,o,e) \xrightarrow{-2} (o,o,e,e)$.

(vii) P1 removes a heap of size 3, then m'_1, m'_3 and m'_4 are even, m'_2 is odd.

i.e. $(e,o,o,e) \xrightarrow{-3} (e,o,e,e)$.

(viii) P1 removes one stone from a heap of size 4, then m'_1 and m'_3 are even, m'_2 and m'_4 are odd. i.e. $(e,o,o,e) \xrightarrow{-1} (e,o,e,o)$.

(ix) P1 removes two stones from a heap of size 4, then m'_1 and m'_2 are even, m'_3 and m'_4 are odd. i.e. $(e,o,o,e)\xrightarrow{-2}(e,e,o,o)$.

(x) P1 removes three stones from a heap of size 4, then m'_1, m'_2, m'_3 and m'_4 are odd. i.e. $(e,o,o,e)\xrightarrow{-3}(o,o,o,o)$.

(xi) P1 removes a heap of size 4, then m'_1 is even, m'_2, m'_3 and m'_4 are odd. i.e. $(e,o,o,e)\xrightarrow{-4}(e,o,o,o)$.

This leaves an N-position of Nim for P2 in (i) while in all other cases, since $n-4 \le n' < n$ and $m'_1 + m'_3 \ge 2$, (m'_1, m'_2, m'_3, m'_4) is not (e,o,o,e) or (o,e,e,e), this leaves an N-position for P2 to play on. Hence (e,o,o,e) is P-position.

Next, we show that the remaining cases are N-positions.

(i) If m_1, m_2, m_3 and m_4 are even, P1 uses the pass option, then m'_1, m'_2, m'_3 and m'_4 are even. i.e. $(e,e,e,e)\xrightarrow{pass}(e,e,e,e)$.

(ii) If m_1, m_2 and m_3 are even, m_4 is odd, P1 removes three stones from a heap of size 4, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(e,e,e,o)\xrightarrow{-3}(o,e,e,e)$.

(iii) If m_1, m_2 and m_4 are even, m_3 is odd, P1 removes two stones from a heap of size 3, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(e,e,o,e)\xrightarrow{-2}(o,e,e,e)$.

(iv) If m_1, m_3 and m_4 are even, m_2 is odd, P1 removes one stone from a heap of size 2, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(e,o,e,e)\xrightarrow{-1}(o,e,e,e)$.

(v) If m_1 and m_4 are odd, m_2 and m_3 are even, P1 removes a heap of size 4, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,e,e,o)\xrightarrow{-4}(o,e,e,e)$.

(vi) If m_1 and m_2 are even, m_3 and m_4 are odd, P1 removes two stones from a heap of size 4, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(e,e,o,o) \xrightarrow{-2} (e,o,o,e)$.

(vii) If m_1 and m_3 are even, m_2 and m_4 are odd, P1 removes one stone from a heap of size 4, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(e,o,e,o) \xrightarrow{-1} (e,o,o,e)$.

(viii) If m_1, m_2 and m_3 are odd, m_4 is even, P1 uses the pass option, then m'_1, m'_2 and m'_3 are odd, m'_4 is even. i.e. $(o,o,o,e) \xrightarrow{pass} (o,o,o,e)$.

(ix) If m_1, m_2 and m_4 are odd, m_3 is even, P1 removes two stones from a heap of size 4, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,o,e,o) \xrightarrow{-2} (o,e,e,e)$.

(x) If m_1, m_3 and m_4 are odd, m_2 is even, P1 removes one stone from a heap of size 4, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,e,o,o) \xrightarrow{-1} (o,e,e,e)$.

(xi) If m_1 and m_2 are odd, m_3 and m_4 are even, P1 removes a heap of size 2, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,o,e,e) \xrightarrow{-2} (o,e,e,e)$.

(xii) If m_1 is even, m_2, m_3 and m_4 are odd, P1 removes a heap of size 4, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(e,o,o,o) \xrightarrow{-4} (e,o,o,e)$.

(xiii) If m_1, m_2, m_3 and m_4 are odd, P1 removes three stones from a heap of size 4, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(o,o,o,o) \xrightarrow{-3} (e,o,o,e)$.

(xiv) If m_1 and m_3 are odd, m_2 and m_4 are even and $m_2 \geq 2$, P1 removes one stone from a heap of size 2, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(o,e,o,e) \xrightarrow{-1} (e,o,o,e)$.

(xv) If m_1 and m_3 are odd, m_2 and m_4 are even and $m_1 + m_3 \geq 3$,P1 removes a heap of size 3, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,e,o,e) \xrightarrow{-3} (o,e,e,e)$.

This leaves a P-position of Nim for P2 in (i) and (viii). In (ii) to (v), (ix) to (xi) and (xv), since $n-4 \le n' < n$, $m'_1 + m'_3 \ge 2$ and (m'_1, m'_2, m'_3, m'_4) is (o,e,e,e), this leaves a P-position for P2 to play on. In (vi) and (viii), (xii) to (xiv), since $n-4 \le n' < n$, $m'_1 + m'_3 \ge 2$ and (m'_1, m'_2, m'_3, m'_4) is (e,o,o,e), this leaves a P-position for P2 to play on. Therefore, these last ten cases are N-positions. Finally, by lemma 2, the remaining case $(1,0,1,e)$ is an N-position.

Consider $n = 20$ and $m_2 - m_4 \ge 4$.

If $m_1 + m_3 \ge 2$ and $m_2 - m_4 \ge 4$, it is proved that this is a P-position if and only if (m_1, m_2, m_3, m_4) is (o,e,e,e) or (e,o,o,e). Therefore, we have to show that this is a P-position if and only if (m_1, m_2, m_3, m_4) is (o,e,e,e) or (e,o,o,e) for $m_1 + m_3 < 2$ and $m_2 - m_4 \ge 4$.

If (m_1, m_2, m_3, m_4) is (o,e,e,e), $m_1 + m_3 < 2$ and $m_2 - m_4 \ge 4$ and

(i) P1 uses the pass option, then m'_1 is odd, m'_2, m'_3 and m'_4 are even.

i.e. $(1,e,0,e) \xrightarrow{pass} (1,e,0,e)$.

(ii) P1 removes a heap of size 1, then m'_1, m'_2, m'_3 and m'_4 are even.

i.e. $(1,e,0,e) \xrightarrow{-1} (0,e,0,e)$.

(iii) P1 removes one stone from a heap of size 2, then m'_1, m'_3 and m'_4 are even, m'_2 is odd.

i.e. $(1,e,0,e) \xrightarrow{-1} (2,o,0,e)$.

(iv) P1 removes a heap of size 2, then m'_1 and m'_2 are odd, m'_3 and m'_4 are even.

i.e. $(1,e,0,e) \xrightarrow{-2} (1,o,0,e)$.

(v) P1 removes one stone from a heap of size 4, then m'_1, m'_3 and m'_4 are odd, m'_2 is even.

i.e. $(1, e, 0, e) \xrightarrow{-1} (1, e, 1, o)$.

(vi) P1 removes two stones from a heap of size 4, then m'_1, m'_2 and m'_4 are odd, m'_3 is even.

i.e. $(1, e, 0, e) \xrightarrow{-2} (1, o, 0, o)$.

(vii) P1 removes three stones from a heap of size 4, then m'_1, m'_2 and m'_3 are even, m'_4 is odd.

i.e. $(1, e, 0, e) \xrightarrow{-3} (2, e, 0, o)$.

(viii) P1 removes a heap of size 4, then m'_1 and m'_4 are odd, m'_2 and m'_3 are even.

i.e. $(1, e, 0, e) \xrightarrow{-4} (1, e, 0, o)$.

This leaves an N-position of Nim for P2 in (i). In (ii), (vi) and (viii), since $n - 4 \le n' < n$, $m'_2 - m'_4 \ge 4$, (m_1, m_2, m_3, m_4) is not (o, e, e, e) or (e, o, o, e), this leaves an N-position for P2 to play on. In (iii), (v) and (vii), since $n - 4 \le n' < n$, $m'_1 + m'_3 \ge 2$, (m_1, m_2, m_3, m_4) is not (o, e, e, e) or (e, o, o, e), this leaves an N-position for P2 to play on. In (iv), if $m_2 - m_4 = 4$, by Lemma 2, this leaves an N-position for P2. If $m_2 - m_4 > 4$, this leaves an N-position for P2 since $n - 4 \le n' < n$, $m'_2 - m'_4 \ge 4$, (m_1, m_2, m_3, m_4) is not (o, e, e, e) or (e, o, o, e). Hence, (o, e, e, e) is a P-position.

If (m_1, m_2, m_3, m_4) is (e, o, o, e), $m_1 + m_3 < 2$ and $m_2 - m_4 \ge 4$ and

(i) P1 uses the pass option, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd.

i.e. $(0, o, 1, e) \xrightarrow{pass} (0, o, 1, e)$.

(ii) P1 removes one stone from a heap of size 2, then m'_1 and m'_3 are odd, m'_2 and m'_4 are even. i.e. $(0, o, 1, e) \xrightarrow{-1} (1, e, 1, e)$.

(iii) P1 removes a heap of size 2, then m'_1, m'_2 and m'_4 are even, m'_3 is odd.

i.e. $(0,o,1,e)\xrightarrow{-2}(0,e,1,e)$.

(iv) P1 removes one stone from a heap of size 3, then m'_1, m'_2, m'_3 and m'_4 are even.

i.e. $(0,o,1,e)\xrightarrow{-1}(0,e,0,e)$.

(v) P1 removes two stones from a heap of size 3, then m'_1 and m'_2 are odd, m'_3 and m'_4 are even. i.e. $(0,o,1,e)\xrightarrow{-2}(1,o,0,e)$.

(vi) P1 removes a heap of size 3, then m'_1, m'_3 and m'_4 are even, m'_2 is odd.

i.e. $(0,o,1,e)\xrightarrow{-3}(0,o,0,e)$.

(vii) P1 removes one stone from a heap of size 4, then m'_1 and m'_3 are even, m'_2 and m'_4 are odd. i.e. $(0,o,1,e)\xrightarrow{-1}(0,o,2,o)$.

(viii) P1 removes two stones from a heap of size 4, then m'_1 and m'_2 are even, m'_3 and m'_4 are odd. i.e. $(0,o,1,e)\xrightarrow{-2}(0,e,1,o)$.

(ix) P1 removes three stones from a heap of size 4, then m'_1, m'_2, m'_3 and m'_4 are odd.

i.e. $(0,o,1,e)\xrightarrow{-3}(1,o,1,o)$.

(x) P1 removes a heap of size 4, then m'_1 is even, m'_2, m'_3 and m'_4 are odd.

i.e. $(0,o,1,e)\xrightarrow{-4}(0,o,1,o)$.

This leaves an N-position of Nim for P2 in (i). In (ii), (vii) and (ix), since $n-4\le n'<n$, $m'_1+m'_3\ge 2$ and (m'_1,m'_2,m'_3,m'_4) is not (o,e,e,e) or (e,o,o,e), this leaves an N-position for P2 to play on. In (iii) to (vi), (viii) and (x), since $n-4\le n'<n$, $m'_2-m'_4\ge 4$, (m'_1,m'_2,m'_3,m'_4) is not (o,e,e,e) or (e,o,o,e), this leaves an N-position for P2 to play on. Hence, (e,o,o,e) is a P-position.

Next, we show that the remaining cases are N-positions.

(i) If m_1, m_2, m_3 and m_4 are even, P1 uses the pass option, then m'_1, m'_2, m'_3 and m'_4 are even. i.e. $(e,e,e,e)\xrightarrow{pass}(e,e,e,e)$.

(ii) If m_1, m_2 and m_3 are even, m_4 is odd, P1 removes three stones from a heap of size 4, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(e,e,e,o)\xrightarrow{-3}(o,e,e,e)$.

(iii) If m_1, m_2 and m_4 are even, m_3 is odd, P1 removes two stones from a heap of size 3, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(e,e,o,e)\xrightarrow{-2}(o,e,e,e)$.

(iv) If m_1, m_3 and m_4 are even, m_2 is odd, P1 removes one stone from a heap of size 2, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(e,o,e,e)\xrightarrow{-1}(o,e,e,e)$.

(v) If m_1 and m_4 are odd, m_2 and m_3 are even, P1 removes a heap of size 4, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,e,e,o)\xrightarrow{-4}(o,e,e,e)$.

(vi) If m_1 and m_2 are even, m_3 and m_4 are odd, P1 removes two stones from a heap of size 4, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(e,e,o,o)\xrightarrow{-2}(e,o,o,e)$.

(vii) If m_1 and m_3 are even, m_2 and m_4 are odd, P1 removes one stone from a heap of size 4, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(e,o,e,o)\xrightarrow{-1}(e,o,o,e)$.

(viii) If m_1, m_2 and m_4 are odd, m_3 is even, P1 removes two stones from a heap of size 4, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,o,e,o)\xrightarrow{-2}(o,e,e,e)$.

(ix) If m_1 and m_2 are odd, m_3 and m_4 are even, P1 removes a heap of size 2, then m'_1 is odd, m'_2, m'_3 and m'_4 are even. i.e. $(o,o,e,e)\xrightarrow{-2}(o,e,e,e)$.

(x) If m_1 is even, m_2, m_3 and m_4 are odd, P1 removes a heap of size 4, then m'_1 and m'_4 are even, m'_2 and m'_3 are odd. i.e. $(e,o,o,o)\xrightarrow{-4}(e,o,o,e)$.

This leaves a P-position of Nim for P2 in (i). In all other cases, since $n-4 \le n' < n$, $m'_2 - m'_4 \ge 4$, (m'_1, m'_2, m'_3, m'_4) is (o,e,e,e) or (e,o,o,e), this leaves a P-position for P2 to play on. Hence, these last ten cases are N-positions.

Therefore, the claim holds for $n = 20$. By induction on the total number of stones, the claim is established.

Conclusion

Unlike the previous result on Nim* played on heaps of sizes at most three, the game positions of Nim* played on heaps of sizes at most four are impossible to locate by using Nim-sum with restriction on the total number of stones. Irregularities are most likely continue when the sum of number of heaps of sizes 1 and 3 is less than 2. Therefore, change of strategy is needed. It is believed that Nim-sum will continue to play an important role in further analysis of Nim* while the importance of the restriction on the total number of stones will be weaken.

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Appendix I: All the possibilities for $1 \le n \le 19$ of Nim* played on heaps of sizes at most four

Table 3: All the possibilities for $1 \le n \le 19$ of Nim* played on heaps of sizes at most four.

n	m_1	m_2	m_3	m_4	Position
1	1	0	0	0	N
2	2	0	0	0	N
	0	1	0	0	N
3	3	0	0	0	P
	1	1	0	0	P
	0	0	1	0	N
4	4	0	0	0	N
	2	1	0	0	N
	1	0	1	0	N
	0	2	0	0	N
	0	0	0	1	N
5	5	0	0	0	N
	3	1	0	0	N
	2	0	1	0	N
	1	2	0	0	N
	1	0	0	1	N
	0	1	1	0	N
6	6	0	0	0	N
	4	1	0	0	N

	3	0	1	0	N
	2	2	0	0	N
	2	0	0	1	N
	0	3	0	0	P
	0	1	0	1	N
7	7	0	0	0	P
	5	1	0	0	N
	4	0	1	0	N
	3	2	0	0	P
	3	0	0	1	N
	2	1	1	0	P
	1	3	0	0	N
	1	1	0	1	N
	1	0	2	0	P
	0	2	1	0	N
	0	0	1	1	P
8	8	0	0	0	N
	6	1	0	0	N
	5	0	1	0	N
	4	2	0	0	N
	4	0	0	1	N
	3	1	1	0	N

	2	3	0	0	N
	2	1	0	1	N
	2	0	2	0	N
	1	2	1	0	N
	1	0	1	1	N
	0	2	0	1	N
	0	0	0	2	N
	9	0	0	0	P
	7	1	0	0	N
	6	0	1	0	N
	5	2	0	0	P
	5	0	0	1	N
	4	1	1	0	N
	3	1	0	1	N
9	3	0	2	0	P
	2	2	1	0	N
	2	0	1	1	N
	1	2	0	1	P
	1	0	0	2	P
	0	3	1	0	N
	0	1	1	1	N
	0	0	3	0	N

10	10	0	0	0	N
	8	1	0	0	N
	7	0	1	0	N
	6	2	0	0	N
	6	0	0	1	N
	5	1	1	0	N
	4	1	0	1	N
	3	0	1	1	N
	2	2	0	1	N
	2	0	0	2	N
	1	3	1	0	N
	1	1	1	1	N
	0	5	0	0	N
	0	3	0	1	N
	0	1	0	2	N
	0	0	2	1	N
11	11	0	0	0	P
	9	1	0	0	N
	8	0	1	0	N
	7	2	0	0	P
	7	0	0	1	N
	6	1	1	0	P

	5	1	0	1	N
	5	0	2	0	P
	4	2	1	0	N
	4	0	1	1	N
	3	2	0	1	N
	3	0	0	2	P
	2	3	1	0	P
	2	1	1	1	N
	2	0	3	0	N
	1	5	0	0	N
	1	3	0	1	N
	1	1	0	2	N
	1	0	2	1	N
	0	4	1	0	N
	0	2	1	1	N
	0	1	3	0	P
	0	0	1	2	N
12	12	0	0	0	N
	10	1	0	0	N
	9	0	1	0	N
	8	2	0	0	N
	8	0	0	1	N

	7	1	1	0	N
	6	1	0	01	N
	6	0	2	0	N
	5	2	1	0	N
	5	0	1	1	N
	4	2	0	1	N
	4	0	0	2	N
	3	1	1	1	N
	3	0	3	0	N
	2	5	0	0	N
	2	3	0	1	N
	2	1	0	2	N
	2	0	2	1	N
	1	2	1	1	N
	1	1	3	0	N
	1	0	1	2	N
	0	6	0	0	N
	0	4	0	1	N
	0	3	2	0	N
	0	2	0	2	N
	0	1	2	1	N
	0	0	0	3	N

13	13	0	0	0	P
	11	1	0	0	N
	10	0	1	0	N
	9	2	0	0	P
	9	0	0	1	N
	8	1	1	0	P
	7	1	0	1	N
	7	0	2	0	P
	6	0	1	1	N
	5	2	0	1	N
	5	0	0	2	P
	4	1	1	1	N
	4	0	3	0	N
	3	5	0	0	N
	3	3	0	1	N
	3	1	0	2	N
	3	0	2	1	N
	2	2	1	1	N
	2	1	3	0	P
	2	0	1	2	N
	1	6	0	0	P
	1	4	0	1	N

	1	2	0	2	N
	1	1	2	1	N
	1	0	4	0	P
	1	0	0	3	N
	0	5	1	0	P
	0	3	1	1	P
	0	1	1	2	P
	0	0	3	1	N
14	14	0	0	0	N
	12	1	0	0	N
	11	0	1	0	N
	10	2	0	0	N
	10	0	0	1	N
	9	1	1	0	N
	8	3	0	0	N
	8	1	0	1	N
	7	2	1	0	N
	7	0	1	1	N
	6	4	0	0	N
	6	2	0	1	N
	6	1	2	0	N
	6	0	0	2	N

	5	3	1	0	N
	5	1	1	1	N
	5	0	3	0	N
	4	5	0	0	N
	4	3	0	1	N
	4	2	2	0	N
	4	1	0	2	N
	4	0	2	1	N
	3	4	1	0	N
	3	2	1	1	N
	3	1	3	0	N
	3	0	1	2	N
	2	6	0	0	N
	2	4	0	1	N
	2	2	0	2	N
	2	1	2	1	N
	2	0	4	0	N
	2	0	0	3	N
	1	5	1	0	N
	1	3	1	1	N
	1	2	3	0	N
	1	1	1	2	N

	1	0	3	1	N
	0	7	0	0	N
	0	5	0	1	N
	0	4	2	0	N
	0	3	0	2	N
	0	2	2	1	N
	0	1	4	0	N
	0	1	0	3	N
	0	0	2	2	N
15	15	0	0	0	P
	13	1	0	0	N
	12	0	1	0	N
	11	2	0	0	P
	11	0	0	1	N
	10	1	1	0	P
	9	3	0	0	N
	9	1	0	1	N
	8	2	1	0	N
	8	0	1	1	N
	7	4	0	0	P
	7	2	0	1	N
	7	1	2	0	N

	7	0	0	2	P
	6	3	1	0	P
	6	1	1	1	N
	6	0	3	0	N
	5	5	0	0	N
	5	3	0	1	N
	5	2	2	0	P
	5	1	0	2	N
	5	0	2	1	N
	4	4	1	0	N
	4	2	1	1	N
	4	1	3	0	P
	4	0	1	2	N
	3	6	0	0	P
	3	4	0	1	N
	3	3	2	0	N
	3	2	0	2	P
	3	1	2	1	N
	3	0	4	0	P
	3	0	0	3	N
	2	5	1	0	P
	2	3	1	1	N

	2	2	3	0	N
	2	1	1	2	P
	2	0	3	1	N
	1	7	0	0	N
	1	5	0	1	N
	1	4	2	0	P
	1	3	0	2	P
	1	2	2	1	N
	1	1	4	0	N
	1	1	0	3	P
	1	0	2	2	P
	0	6	1	0	N
	0	4	1	1	N
	0	3	3	0	P
	0	2	1	2	N
	0	1	3	1	N
	0	0	5	0	N
	0	0	1	3	N
16	16	0	0	0	N
	14	1	0	0	N
	13	0	1	0	N
	12	2	0	0	N

	12	0	0	1	N
	11	1	1	0	N
	10	1	0	1	N
	10	0	2	0	N
	9	2	1	0	N
	9	0	1	1	N
	8	4	0	0	N
	8	2	0	1	N
	8	1	2	0	N
	8	0	0	2	N
	7	3	1	0	N
	7	1	1	1	N
	7	0	3	0	N
	6	5	0	0	N
	6	3	0	1	N
	6	2	2	0	N
	6	1	0	2	N
	6	0	2	1	N
	5	2	1	1	N
	5	1	3	0	N
	5	0	1	2	N
	4	6	0	0	N

	4	4	0	1	N
	4	3	2	0	N
	4	2	0	2	N
	4	1	2	1	N
	4	0	4	0	N
	4	0	0	3	N
	3	5	1	0	N
	3	3	1	1	N
	3	2	3	0	N
	3	1	1	2	N
	3	0	3	1	N
	2	7	0	0	N
	2	5	0	1	N
	2	4	2	0	N
	2	3	0	2	N
	2	2	2	1	N
	2	1	4	0	N
	2	1	0	3	N
	2	0	2	2	N
	1	6	1	0	N
	1	4	1	1	N
	1	3	3	0	N

	1	2	1	2	N
	1	1	3	1	N
	1	0	5	0	N
	1	0	1	3	N
	0	8	0	0	N
	0	6	0	1	N
	0	5	2	0	N
	0	4	0	2	N
	0	3	2	1	N
	0	2	4	0	N
	0	2	0	3	N
	0	1	2	2	N
	0	0	4	1	N
	0	0	0	4	N
17	17	0	0	0	P
	15	1	0	0	N
	14	0	1	0	N
	13	0	0	1	N
	12	1	1	0	P
	11	3	0	0	N
	11	1	0	1	N
	11	0	2	0	P

	10	2	1	0	N
	10	0	1	1	N
	9	2	0	1	N
	9	1	2	0	N
	9	0	0	2	P
	8	3	1	0	P
	8	1	1	1	N
	8	0	3	0	N
	7	5	0	0	N
	7	3	0	1	N
	7	2	2	0	P
	7	1	0	2	N
	7	0	2	1	N
	6	4	1	0	N
	6	2	1	1	N
	6	1	3	0	P
	6	0	1	2	N
	5	6	0	0	P
	5	4	0	1	N
	5	3	2	1	N
	5	2	0	2	P
	5	1	2	1	N

	5	0	4	0	P
	5	0	0	3	N
	4	5	1	0	P
	4	3	1	1	N
	4	2	3	0	N
	4	1	1	2	P
	4	0	3	1	N
	3	7	0	0	N
	3	5	0	1	N
	3	4	2	0	P
	3	3	0	2	N
	3	2	2	1	N
	3	1	4	0	N
	3	1	0	3	N
	3	0	2	2	P
	2	6	1	0	N
	2	4	1	1	N
	2	3	3	0	P
	2	2	1	2	N
	2	1	3	1	N
	2	0	5	0	N
	2	0	1	3	N

	1	8	0	0	P
	1	6	0	1	N
	1	5	2	0	N
	1	4	0	2	N
	1	3	2	1	N
	1	2	4	0	P
	1	2	0	3	N
	1	1	2	2	N
	1	0	4	1	N
	1	0	0	4	N
	0	7	1	0	P
	0	5	1	1	N
	0	4	3	0	N
	0	3	1	2	N
	0	2	3	1	N
	0	1	4	0	N
	0	1	1	3	N
	0	0	3	2	N
18	18	0	0	0	N
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	15	0	1	0	N
	14	2	0	0	N

	14	0	0	1	N
	13	1	1	0	N
	12	3	0	0	N
	12	1	0	1	N
	11	0	1	1	N
	10	4	0	0	N
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	10	0	0	2	N
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	9	1	1	1	N
	9	0	3	0	N
	8	5	0	0	N
	8	3	0	1	N
	8	2	2	0	N
	8	1	0	2	N
	8	0	2	1	N
	7	2	1	1	N
	7	1	3	0	N
	7	0	1	2	N
	6	6	0	0	N
	6	4	0	1	N

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	6	2	0	2	N
	6	1	2	1	N
	6	0	4	0	N
	6	0	0	3	N
	5	5	1	0	N
	5	3	1	1	N
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	4	1	0	3	N
	4	0	2	2	N
	3	6	1	0	N
	3	4	1	1	N
	3	3	3	0	N
	3	2	1	2	N

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	3	0	5	0	N
	3	0	1	3	N
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	2	6	0	1	N
	2	5	2	0	N
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	1	1	4	0	N
	1	1	1	3	N
	1	0	3	2	N
	0	9	0	0	N

	0	7	0	1	N
	0	6	2	0	N
	0	5	0	2	P
	0	4	2	1	N
	0	3	4	0	N
	0	3	0	3	N
	0	2	2	2	N
	0	1	4	1	N
	0	1	0	4	N
	0	0	2	3	N
19	19	0	0	0	P
	17	1	0	0	N
	16	0	1	0	N
	15	2	0	0	P
	15	0	0	1	N
	13	3	0	0	N
	13	1	0	1	N
	12	2	1	0	N
	12	0	1	1	N
	11	4	0	0	P
	11	2	0	1	N
	11	1	2	0	N

	11	0	0	2	P
	10	3	1	0	P
	10	1	1	1	N
	9	5	0	0	N
	9	3	0	1	N
	9	2	2	0	P
	9	1	0	2	N
	9	0	2	1	N
	8	4	1	0	N
	8	2	1	1	N
	8	1	3	0	P
	8	0	1	2	N
	7	6	0	0	P
	7	4	0	1	N
	7	3	2	0	N
	7	2	0	2	P
	7	1	2	1	N
	7	0	4	0	P
	7	0	0	3	N
	6	5	1	0	P
	6	3	1	1	N
	6	2	3	0	N

	6	1	1	2	P
	6	0	3	1	N
	5	7	0	0	N
	5	5	0	1	N
	5	4	2	0	P
	5	3	0	2	N
	5	2	2	1	N
	5	1	4	0	N
	5	1	0	3	N
	5	0	2	2	P
	4	6	1	0	N
	4	4	1	1	N
	4	3	3	0	P
	4	2	1	2	N
	4	1	3	1	N
	4	0	5	0	N
	4	0	1	3	N
	3	8	0	0	P
	3	6	0	1	N
	3	5	2	0	N
	3	4	0	2	P
	3	3	2	1	N

	3	2	4	0	P
	3	2	0	3	N
	3	1	2	2	N
	3	0	4	1	N
	3	0	0	4	P
	2	7	1	0	P
	2	5	1	1	N
	2	4	3	0	N
	2	3	1	2	P
	2	2	3	1	N
	2	1	4	0	N
	2	1	1	3	N
	2	0	3	2	N
	1	9	0	0	N
	1	7	0	1	N
	1	6	2	0	P
	1	5	0	2	N
	1	4	2	1	N
	1	3	4	0	N
	1	3	0	3	N
	1	2	2	2	P
	1	1	4	1	N

	1	1	0	4	N
	1	0	6	0	P
	1	0	2	3	N
	0	8	1	0	N
	0	6	1	1	N
	0	5	3	0	P
	0	4	1	2	N
	0	3	3	1	N
	0	2	5	0	N
	0	2	1	3	P
	0	1	3	2	P
	0	0	5	1	N
	0	0	1	4	P