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Optimal design of simple step-stress accelerated life tests for one-shot devices under Weibull distributions

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STATEMENT OF ORIGINALITY

I, Hu Xuwen, hereby declare that I am the sole author of the thesis and the material presented in this thesis is my original work except those indicated in the acknowledgement. I further declare that I have followed the University's policies and regulations on Academic Honesty, Copyright and Plagiarism in writing the thesis and no material in this thesis has been submitted for a degree in this or other universities.

ABSTRACT

"One-shot" devices are widely used in many fields nowadays. All of these devices can be used only once and then they will be destroyed extensively, which is called "one-shot" devices. Weibull distribution presents a more flexible model than Exponential distribution, is adopted in this paper. Optimal deign for sample allocation with simple step-stress accelerated life test under a Weibull cumulative exposure distribution will be designed. In this paper we get the information matrix after obtaining derivatives of observed log-likelihood function, and asymptotic covariance matrix of the model parameters in order that optimal design that minimized the asymptotic variance of estimate of the reliability at a mission time of simple step-stress accelerated life test using maximum loglikelihood estimation under normal conditions in terms of one decided variable, sample allocation. Also, a procedure that determines the sample allocation with the termination time, inspection time, stress level and normal condition are given. Simulation studies are conducted to show the process of optimal design is reliable.

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Introduction

Nowadays, more and more high-technique devices are used in many fields and give us a safe and convenient daily life. In our life the airbags of automobile reduce damage to the minimum when having an accident; battery could provide electrical power to devices in a limited condition, and rockets help us to explore the outer space. All of these devices have the same characteristic, that is, just only can be used once and then they have to be rebuilt or destroyed extensively. These machines are called "one-shot" devices. Although these products give us so many conveniences, their reliability still likes a "secret". If the reliability at a mission time could be obtained in their life, producers and experiment conductor can make better use of "one-shot" devices. Avoiding to infinite labor and expenditure continuously is spent on these devices, practitioners devised a method to observe the failure product more quickly than they would under normal conditions. Over the years, the terminology "accelerated life test" has been used to describe all such method. In accelerated life testing (ALT), devices are experienced under a high-level condition to shorten the failure time. Typically, stresses are temperature, vibration, humidity, voltages currents, cycling rate. Escobar and Meeker (2006) completely reviewed several accelerated test models planning, statistical models for acceleration and methodologies of parameter estimation.

Trevisanello et al. (2008) studied the 1-W high brightness light-emitting diodes (LEDs) to several stress conditions in short-term accelerated life test. Zheng et al. (2018) investigated the aging performance of grease-based magnetorheological fluids (G-MRFs) at elevated temperature in accelerated life test.

There are many types of accelerated life test. Constant life test (CSALT) is one of the accelerated life tests, which devices only run under one stress level in the whole process either normal condition or pre-specified condition. Yang (1994) explored an accelerated life test plan with four-level constant stresses in different censoring time. Chen et al. (2012) designed a constant stress level accelerated life test of type-I censored data with two different constant stress level on non-rectangular test region. Sari et al. (2009) carried out a model for actual LED lamps with bivariate constant stress model. Later, Balakrishnan and Ling (2014) discussed optimal sample allocation, inspection frequency and number of inspections at each stress level by using asymptotic variance approach with the constant-stress accelerated life test under Weibull distribution. Another type of accelerated life test is step-stress accelerated life test (SALT) also called as partially accelerated life test, in which only stress level was changed at pre-specific time (Gouno & Balakrishnan, 2005). Besides, progress-stress accelerated life test is similar to the step-stress accelerated life test, but the stress levels in the progress-stress accelerated life is a continuous (progressive) function. Yin and Sheng (1987) derived the distribution of mean lifetime with progress-stress accelerated life test in which stress level is proportional to time under Weibull distribution or exponential distribution. Subsequently, Lin and Fei (1991) proposed a methodology to estimate mean lifetime at a normal condition in a progressive-stress accelerated life test. Wang and Fei (2004) presented estimated lifetime in tampered failure rate model with progressive stress accelerated life test under Weibull distribution and conducted a Monte-Carlo simulation. More specifically, simple step-stress accelerated life test (SSALTs) is a particular step-stress accelerated life test. In this testing, devices have to run under original stress at the first stage and then only change the stress to the other condition once at the same time (Ebrahem, 2012). Zhao and Elsayed (2005) proposed a general accelerated life model and likelihood formulation for accelerated life test for Weibull distribution and lognormal distribution.

Compare to the constant-stress model, simple step-stress accelerated life test needs less sample size and could save more time and expenditure in the experiment (Ling, 2019). Many scholars showed their strong interests and have conducted kinds of research for simple step-stress accelerated life test. Bhattacharyya and Soejoeti (1989) presented a statistical model and its properties, inferred lifetime and a regression structure in simple step-stress accelerated life test. Alhadeed and Yang (2005) introduced the optimal times of changing stress level for the simple step-stress test plan, predicted the lifetime of products and provided a range of parameters value under log-normal distribution. Fard and Li (2009) extended the results of Alhadeed and Yang (2005) to Weibull distribution for failure data time by using the Khamis-Higgins (K-H) model. Kateri and Balakrishnan (2008) conducted a study to infer the parameters of a simple step-stress model under Weibull distribution and interval estimation for Type-II censoring data. Later, Sha and Pan (2014) did a similar research and further compared the difference between Weibull cumulative model and Weibull proportional hazard model for simple step-stress accelerated life test.

Although simple step-stress life test is a prevalent method used in many kinds of literatures, the different model and distribution would lead to a different result even using simple step-stress accelerated life test. There are three conventional models: tampered failure-rate model (Bhattacharyya & Soejeoti, 1989), tampered random variable model (DeGroot & Goel, 1979) and cumulative exposure model (Nelson, 1980; Nelson, 1990). Introduced by Nelson (1980), specimens only depend on the current stress and current cumulative fraction failed in the remaining lifetime regardless of accumulation history. Additionally, cumulative exposure (CE) model is widely adopted in engineering reliability studies. Alhadeed and Yang (2005) designed an optimal time of changing stress level with cumulative exposure model under log-normal distribution. Fard and Li (2009) also used cumulative exposure model under exponential distribution to estimate the reliability at mission time. Dorp and Mazzuchi (2005) analyzed varying-stress reliability tests by using Bayesian approaching under Weibull cumulative exposure model.

Besides, gamma distribution (Balakrishnan & Ling, 2014), exponential distribution (Balakrishnan & Ling, 2012; Balakrishnan *et al.,* 2015), Weibull distribution (Bai & Kim, 1993; Balakrishnan & Mitra, 2012; Balakrishnan & Ling, 2013), log-normal distribution (Alhadeed & Yang, 2005;), log-logistic distribution (Ebrahem, 2012), Birnbaum – Saunders distribution (Sun & Shi, 2016) and lognormal distribution are some of the distributions commonly used for engineering reliability analysis. Lognormal distribution usually models some failure data caused by corrosion and chemical reactions. Zhao and Elasyed (2005) presented a likelihood function model for the lognormal distribution. Balakrishnan and Ling (2014) analyzed left-or-right censored data under gamma distribution and obtained the asymptotic confident interval to estimate the life distribution under normal condition. Also, Balakrishnan *et al.* (2012) developed Expectation Maximization (EM) algorithm to estimate the model parameters and furthermore compare to estimation obtained by Inequality Constrained Least Squares (ICLS) under exponential distribution. The exponential distribution is a popular distribution used in practice, and the hazard function is constant (does not depend on time). Ling (2019) presented an EM algorithm to conduct an optimal design of sample allocation, stress level, inspection time under exponential distribution. Ebrahem (2012) designed optimum times of changing stress level plans under the log-logistic model and used maximum likelihood estimation to construct interval for estimated parameters. Birnbaum-Saunders distribution is a model for the number of cycles necessary to force a fatigue crack to grow to a critical size derived by Birnbaum and Saunders (1969). Sun and Shi (2016) used Bayesian method to obtain parameter focusing on Type II censoring data under Birnbaum–Saunders distribution.

Weibull distribution is a continuous probability distribution which used to describe various situations of observed failure of components, commonly applied in engineering studies to assess the reliability of products, lifetime and model failure time. Bai and Kim (1993) used monograph to find the optimal stress and time for Type-I censoring data in the design under Weibull distribution. Balakrishnan and Mitra (2008) developed EM algorithm for left truncated and right censored data under Weibull distribution and constructed the confidence interval for estimated parameters. Kateri and Balakrishnan (2008) did a similar research but focused on Type II censoring data. Balakrishnan and Ling (2013) also presented EM algorithm to estimate the parameters and constructed confidence interval but based on binary data under Weibull distribution. Nandi and Dewan (2010) estimated parameters of Maximum likelihood estimation of Marshall- Olkin Bivariate Weibull distribution and test the performance of proposed estimators through simulation studies. Furthermore, Weibull distribution has a unique linkage with exponential distribution. Especially when the shape parameter equals to one which means the hazard rate is constant over time, Weibull distribution would reduce to the exponential distribution. In this case, the performance of the devices does not depend on time. When Weibull distribution has an increasing hazard function, it presents the failure rate would increase as time goes on which is a feature most devices have. Also, if the older machines have a better performance, shape parameter of Weibull distribution (hazard rate) would be smaller than one. Weibull distribution has certain analytical advantages over the exponential distribution and shows its flexibility to fit various traits of devices. Therefore, the Weibull distribution would be adopted in this research and also discuss three situations of shape parameter separately.

When performing a reliability analysis of these devices, the exact lifetime is very difficult to collect due to incomplete data. Truncation and censoring are two distinct phenomena cause the incomplete data to happen. Left censoring data are the specimens have already failed before the study started. Right censoring occurs when data failure after the study ends or the subjects quit the study before it ends. Meanwhile, there are two specific types of right censoring: Type-I censoring and Type-II censoring. In Type-I censoring, the time of study is fixed and in the whole process the number of failures is random. Study continues to run until a pre-specified number of failures occur, the study time is arbitrary, which is called Type-II censoring. Besides, right truncation occurs that the whole sample of specimens is failed before the study starts. In left truncation, we only observe those lifetimes of those individuals exceed the truncation time. In practical, the difference between the censored data and truncated data is that the number of censored data is known and the number of truncated are unknown or undiscovered. In this paper, the data we considered is neither Type-I censoring data nor Type-II censoring data. Left-truncated and right-censoring data will be considered in this paper. Also, for those who study the "one-shot" devices under Weibull cumulative exposure distribution, most of them use EM algorithm to estimate the mean lifetime at normal condition, but reliability estimation is scarce. Also, most literatures in the simulation studies consider several variables in terms of stress level, sample allocation and inspection time and so on. Detailed considering sole variable is scarce in the engineering research. Therefore, simple step-stress accelerated life test under Weibull distribution would be adopted and then estimate the reliability under normal condition at a mission time in terms of one decision variable, sample allocation.

Fisher (1922) proposed the method of Maximum likelihood estimation in his influential paper which has become one of the most important tools to estimate parameters for statisticians. Maximum likelihood estimation (MLE) is a statistical method to estimate the parameters in the mathematical model especially for log-location-scale distribution (Kleinbaum & Klein, 2010). Maximum likelihood estimation requires less restriction of any kinds on the characteristics of the independent variables compare to commonly used Least square estimation (LSE). The variables in maximum likelihood estimation could be nominal, ordinal or interval. Yang (1994) found the maximum likelihood to minimize the asymptotic variance of the mean lifetime at pre-specified stress and inspection time. Ebrahem (2012) used maximum likelihood estimation to investigate the parameters from the Fisher information matrix and minimized the asymptotic variance of the reliability. Kateri and Balakrishnan (2008) also used the same method to obtain the minimum asymptotic variance and further analyzed the bias and mean square error of estimated parameters. Scheike and Sun (2007) applied the maximum likelihood estimation to develop EM algorithm to determine tied survival data. By using the experience of these literatures for reference, maximum likelihood estimation will be used in this paper to estimate the parameter in Weibull distribution, and the asymptotic variance of the reliability under normal condition at mission time would be studied.

Optimal design, the selection of the best alternative is the phase of design optimization (Papalambros & Wilde, 2000) where the design describes as a system defined by designed variables, parameters and constants. Meanwhile, there are many optimality information-based criteria; possible criteria are G-, D-, A-, E-, and I-optimality criteria. Smith (1918) introduced the G-optimality is a "prediction criterion" and minimized predicted variance. A-optimality principle was carried out by the Chernoff (1953) to minimize the trace of the inverse of information matrix which minimize the average variance of the estimates. Ehrenfeld (1955) presented E-optimality criteria to maximize the minimum eigenvalue of the information matrix. In many engineering reliability research, the optimal design is very common and has attached great attention from scholars. Bai and Kim (1993) designed an optimal plan of the time to change stress. Fard and Li (2009) also constructed an optimal design with minimizing asymptotic variance of reliability at mission time. Balakrishnan and Ling (2014) also designed optimal plan with constant - stress accelerated life test. Ling (2019) presented simple-step stress accelerated life test plans for one-shot devices. Optimal design would be conducted in this research, set sample allocation as only one determined variable while stress level and inspection time are given and then the procedure of optimal plan also will be explained.

The rest of the article is organized as follows. Section II presents a simple step-stress model under Weibull cumulative exposure distribution and corresponding cumulative hazard function, reliability function and probability density function. In Section III, we provide a detailed process to find asymptotic variance of maximum likelihood estimation based on observed Fisher information matrix and find the optimal sample allocation which minimize the standard error of the reliability. A few simulation studies with different shape parameter in several settings will be conducted in chapter IV. Some limitations will be given in chapter V. Finally, we will make some concluding remarks of this paper in chapter VI.

Model and Description

Suppose $0< IT_1 < IT_2$, $K_1 < K_2$, and $x_1 < x_2$. First, put all K devices to the stress level x_1. And K_1 devices are tested at pre-specified time IT_1, record the number of failures n_1. Next, the remaining devices $K_2 = K - K_1$ exposed to the stress level x_2 under pre-specified inspection time IT_2, and also recorded the number of failures n_2. The data of one-shot devices under simple step-stress accelerated life test can be summarized as Table 1. Given the one-shot devices data $z = \{IT_i, K_i, n_i, x_i, i = 1,2\}$.

Stage	Inspection time	Tested devices	failures	Stress level
1	IT_1	K_1	n_1	x_1
2	IT_2	$K_2 = K - K_1$	n_2	x_2

Table 1：data of one-shot devices under SSALTs

Assume the reliability of devices in this research that follows Weibull distribution. Consequently, cumulative hazard function $H(t)$ is the integral of the hazard function. The hazard function is the ratio of the probability density $f(t)$ to the reliability function $R(t)$. The hazard function describes the instantaneous failure rate at any time rather than a probability. Reliability function (or complementary cumulative distribution function) $R(t)$ is the probability that a unit survives beyond time x and the sum of the reliability function and the cumulation distribution function $F(t)$ in always one. Then, probability density function $f(t)$ is the probability that variable takes the value which is the value x and cumulation distribution function $F(t)$ is integral of probability density function $f(t)$.

Cumulative hazard function, reliability function and probability density function under Weibull distribution would be derived as

$$H(t) = \begin{cases} \left(\frac{t}{\alpha_1}\right)^{\mathrm{k}}, & 0 < t \leq IT_1 \\ \left(\frac{\frac{\alpha_2}{\alpha_1}IT_1 + t - IT_1}{\alpha_2}\right)^{\mathrm{k}}, & t > IT_1 \end{cases},$$

$$R(t) = \exp\left(-\mathrm{H(t)}\right) = \begin{cases} \exp(-\left(\frac{t}{\alpha_1}\right)^{\mathrm{k}}), & 0 < t \leq IT_1 \\ \exp\left(-\left(\frac{\frac{\alpha_2}{\alpha_1}IT_1 + t - IT_1}{\alpha_2}\right)^{\mathrm{k}}\right), & t > IT_1 \end{cases},$$

And

$$f(t) = -R'(t)$$

$$= \begin{cases} \dfrac{k}{{\alpha_1}^{\kappa}}(t)^{k-1}\exp(-\left(\dfrac{t}{\alpha_1}\right)^{\mathrm{k}}), & 0 < t \leq IT_1 \\ \dfrac{k}{{\alpha_2}^{\kappa}}(\dfrac{\alpha_2}{\alpha_1}IT_1 + t - IT_1)^{k-1}\exp\left(-\left(\dfrac{\dfrac{\alpha_2}{\alpha_1}IT_1 + t - IT_1}{\alpha_2}\right)^{\mathrm{k}}\right), & t > IT_1 \end{cases}$$

Where α_1 and α_2 are the scale parameters at first stage and second stage respectively, and $\alpha_1 > 0$, $\alpha_2 > 0$. Also assumed Weibull scale parameters have a log-linear relationship with stress levels (Wang & Kececioglu, 2000). The log-linear form is commonly used in accelerated life test, the form as

$$\alpha_i = e^{(a_0 + a_1 x_i)}$$

For the easily notate these symbols in the remaining part, we make θ= $\{a_0, a_1, \kappa\}$ as the model parameter to estimate.

Optimal design of Simple-Step Accelerated Life Test

The only determined variable is sample allocation, then presents a procedure to find maximum likelihood estimation of the estimated parameters and discusses how to choose an optimal sample allocation when stress level, inspection time and termination time are given.

Information matrix used to describe the amount of information data provide about an unknown parameter (Lehman & Casella, 1998). Louis (1982) pointed out a technique for computing the information matrix extracted from the Missing Information Principle. The information matrix could be derived when complete information and missing information matrix held. The complete information matrix and missing information matrix as follow,

$$I_{complete} = -E\left[\frac{\partial^2\left(\ell_c(\theta)\right)}{\partial\theta^2}\right]$$

$$I_{missing} = -E\left[\frac{\partial^2(\log(f(t_{ij}|z,\theta)))}{\partial\theta^2}\right]$$

And Information matrix is the difference between complete information matrix and missing information matrix as,

$$I(\theta) = I_{complete} - I_{missing}$$

The aim that developing the information matrix is estimating the standard error of reliability. Balakrishnan and Ling (2013) stated that the information matrix is equivalent to the expectation of the second derivative of the observed log-likelihood function. In the one-shot devices testing data, the observed log-likelihood function is given by

$$\ell(\theta) = \sum_{i=1}^{2} n_i \log\big(1 - R(IT_i;\theta)\big) + (K_i - n_i)\log\left(R(IT_i;\theta)\right) + constant$$

The second-derivative of the observed log-likelihood function is obtained as follow:

$$\frac{\partial^2 \ell(\theta)}{\partial a_p \partial a_q} = \sum_{i=1}^{2} \left(\frac{\partial^2 R(IT_i;\theta)}{\partial a_p \partial a_q}\right)\left(-\frac{n_i}{1 - R(IT_i;\theta)} + \frac{K_i - n_i}{R(IT_i;\theta)}\right)$$

$$- \sum_{i=1}^{2} \left(\frac{\partial R(IT_i;\theta)}{\partial a_p}\right)\left(\frac{\partial R(IT_i;\theta)}{\partial \partial a_q}\right)\left(\frac{n_i}{(1 - R(IT_i;\theta)^2} + \frac{K_i - n_i}{\big(R(IT_i;\theta)\big)^2}\right)$$

$$\frac{\partial^2 \ell(\theta)}{\partial a_i \partial k} = \sum_{i=1}^{2} \left(\frac{\partial^2 R(IT_i;\theta)}{\partial a_i \partial k}\right)\left(-\frac{n_i}{1 - R(IT_i;\theta)} + \frac{K_i - n_i}{R(IT_i;\theta)}\right)$$

$$- \sum_{i=1}^{2} \left(\frac{\partial R(IT_i;\theta)}{\partial a_i}\right)\left(\frac{\partial R(IT_i;\theta)}{\partial k}\right)\left(\frac{n_i}{(1 - R(IT_i;\theta)^2} + \frac{K_i - n_i}{\big(R(IT_i;\theta)\big)^2}\right),$$

where

$$\frac{\partial R(IT_i;\theta)}{\partial a_0} = d_{i0}R(IT_i)\mathrm{k}$$

$$\frac{\partial R(IT_i;\theta)}{\partial a_1} = d_{i1}R(IT_i)\mathrm{k}$$

$$\frac{\partial R(IT_1;\theta)}{\partial k} = -d_{10}R(IT_i)\ln\left(\frac{IT_1}{\alpha_1}\right)$$

$$\frac{\partial R(IT_2;\theta)}{\partial k} = -d_{20}R(IT_i)\ln\left(\frac{IT_1}{\alpha_1}+\frac{IT_2-IT_1}{\alpha_2}\right)$$

$$d_{1m} = \left(\frac{IT_1}{\alpha_1}\right)^{\kappa-1}\left(\frac{IT_1}{\alpha_1}\right)x_1{}^m$$

$$d_{2m} = \left(\frac{IT_1}{\alpha_1}+\frac{IT_2-IT_1}{\alpha_2}\right)^{\kappa-1}\left(\frac{IT_1}{\alpha_1}x_1{}^m+\frac{IT_2-IT_1}{\alpha_2}x_2{}^m\right)$$

Furthermore, the Fisher information matrix is the negation of the expectation of second derivatives of the log-likelihood function,

$$I(\theta) = -E\left[\frac{\partial^2\ell(\theta)}{\partial\theta_i\partial\theta_j}\right]$$

Through inverting the information matrix could be gained the asymptotic covariance matrix of the MLEs of the model parameters. The variance of the MLEs of the reliability under normal conditions by using the delta method after obtaining the first-order derivatives of the reliability with different model parameters.

$$I(\theta) = -E\left[\frac{\partial^2 \ell(\theta)}{\partial\theta_i \partial\theta_j}\right] = \sum_{i=1}^{2}\left(\frac{K_i}{R(IT_i)} + \frac{K_i}{1-R(IT_i)}\right)\left(\frac{\partial R(IT_i;\theta)}{\partial a_p}\right)\left(\frac{\partial R(IT_i;\theta)}{\partial a_q}\right),$$

Where p =0,1,2 and q =0,1,2 when $a_2 = k$

At first stage, put the π_1 percentage of all K devices to stress level, and remaining devices $(1-\pi_1)$ of all devices to stress level x_2 at second stage. Let $K_i = K\pi_i$ with $\pi_2 = 1-\pi_1$, $A_i = R(IT_i)^{-1} + (1-R(IT_i))^{-1}$, $X_{ik} = \frac{\partial R(IT_i;\theta)}{\partial a_k}$. Therefore, the fisher information matrix is

$$-E\left[\frac{\partial^2 \ell(\theta)}{\partial\theta_i \partial\theta_j}\right]$$

$$= K\begin{bmatrix} \pi_1 A_1 X_{10}^2 + (1-\pi_1)A_2 X_{20}^2 & \pi_1 A_1 X_{10}X_{11} + (1-\pi_1)A_2 X_{20}X_{21} & \pi_1 A_1 X_{10}X_{12} + (1-\pi_1)A_2 X_{20}X_{22} \\ \pi_1 A_1 X_{10}X_{11} + (1-\pi_1)A_2 X_{20}X_{21} & \pi_1 A_1 X_{11}^2 + (1-\pi_1)A_2 X_{21}^2 & \pi_1 A_1 X_{11}X_{12} + (1-\pi_1)A_2 X_{21}X_{22} \\ \pi_1 A_1 X_{10}X_{12} + (1-\pi_1)A_2 X_{20}X_{22} & \pi_1 A_1 X_{11}X_{12} + (1-\pi_1)A_2 X_{21}X_{22} & \pi_1 A_1 X_{12}^2 + (1-\pi_1)A_2 X_{22}^2 \end{bmatrix}$$

$$= K\begin{bmatrix} r_{00}+s_{00}\pi_1 & r_{10}+s_{10}\pi_1 & r_{20}+s_{20}\pi_1 \\ r_{10}+s_{10}\pi_1 & r_{11}+s_{11}\pi_1 & r_{21}+s_{21}\pi_1 \\ r_{20}+s_{20}\pi_1 & r_{21}+s_{21}\pi_1 & r_{22}+s_{22}\pi_1 \end{bmatrix}$$

Where $r_{kk} = A_2X_{2k}^2$, $r_{mn} = A_2X_{2m}X_{2n}$, $s_{kk} = A_1X_{1k}^2 - A_2X_{2k}^2$, $s_{mn} = A_1X_{1m}X_{1n} - A_2X_{2m}X_{2n}$.

The determinant of information matrix equals to zero. Therefore, the information matrix would reduce to a 2×2 matrix. We assume shape parameter of Weibull distribution k is known, remaining parameters a_0 and a_1 are unknown.

Therefore, the advanced information matrix just remains entries consist of a_0 and a_1 as follow,

$$-E\left[\frac{\partial^2 \ell(\theta)}{\partial\theta_i \partial\theta_j}\right] = K\begin{bmatrix} r_{00}+s_{00}\pi_1 & r_{10}+s_{10}\pi_1 \\ r_{10}+s_{10}\pi_1 & r_{11}+s_{11}\pi_1 \end{bmatrix}$$

The asymptotic covariance matrix of the MLEs of the model parameters is the inverse of the information matrix, it becomes

$$V_\theta = I^{-1}(\theta) = \frac{1}{D}\begin{bmatrix} r_{11}+s_{11}\pi_1 & -(r_{10}+s_{10}\pi_1) \\ -(r_{10}+s_{10}\pi_1) & r_{00}+s_{00}\pi_1 \end{bmatrix},$$

Where $D = K\big((r_{11}+s_{11}\pi_1)(r_{00}+s_{00}\pi_1\big) - (r_{10}+s_{10}\pi_1)^2)$.

The variance of the MLEs of the reliability under the normal operating condition

$$V_{\hat{R}(t)} = P'V_\theta P \ = [d_0 R(t)\mathrm{k} \quad d_1 R(t)\mathrm{k}]\, V_\theta \begin{bmatrix} d_0 R(t)\mathrm{k} \\ d_1 R(t)\mathrm{k} \end{bmatrix}$$

with $d_0 = \left(\frac{t}{\alpha_0}\right)^{\kappa-1}\left(\frac{t}{\alpha_0}\right)$ and $d_1 = \left(\frac{t}{\alpha_0}\right)^{\kappa-1}\left(\frac{t}{\alpha_0}\right)x_0$.

Where P is 2×1 column vector consists of first-order derivatives of the reliability respect to the model parameters a_0, a_1. V_θ is the inverse of the information matrix.

Then,

$$V_{\hat{R}(t)} = P'V_\theta P$$

$$= \frac{R(t)^2 k^2((d_0^2 r_{11} - 2d_0 d_1 r_{10} + d_1^2 r_{00}) + (d_0^2 s_{11} - 2d_0 d_1 s_{10} + d_1^2 s_{00})\pi_1)}{K((r_{11}r_{00} - r_{10}^2) + (r_{11}s_{00} + r_{00}s_{11} - 2r_{10}s_{10})\pi_1 + (s_{00}s_{11} - s_{10}^2)\pi_1^2}$$

$(r_{11}r_{00} - r_{10}^2)$ equals to zero, $(r_{11}s_{00} + r_{00}s_{11} - 2r_{10}s_{10})$ and $(s_{00}s_{11} - s_{10}^2)$ are opposite number. Therefore,

$$V_{\hat{R}(t)}$$

$$= \frac{R(t)^2 k^2((d_0^2 r_{11} - 2d_0 d_1 r_{10} + d_1^2 r_{00}) + (d_0^2 s_{11} - 2d_0 d_1 s_{10} + d_1^2 s_{00})\pi_1)}{K((r_{11}s_{00} + r_{00}s_{11} - 2r_{10}s_{10})\pi_1(1-\pi_1)}$$

As mentioned before, the intention of optimal design is to minimize the standard error of reliability based on sample allocation as follow,

$$\pi_1 = \arg\min_{0<\pi_1<1} \frac{c_1 + c_2\pi_1}{\pi_1(1-\pi_1)} = \arg\min_{0<\pi_1<1} \frac{c_1}{\pi_1} + \frac{c_1 + c_2}{1-\pi_1}$$

Where $c_1 = \frac{R(t)^2 k^2((d_0^2 r_{11} - 2d_0 d_1 r_{10} + d_1^2 r_{00})}{K(r_{11}s_{00} + r_{00}s_{11} - 2r_{10}s_{10})}$ and

$$c_2 = \frac{R(t)^2 k^2(d_0^2 s_{11} - 2d_0 d_1 s_{10} + d_1^2 s_{00})}{K(r_{11}s_{00} + r_{00}s_{11} - 2r_{10}s_{10})}.$$

The variance of reliability would get the minimum value when first derivative $\frac{\partial V_\mu}{\partial \pi_1}$ equals to zero, the value of π_1 is

$$\pi_1 = (1 + \sqrt{\frac{c_1 + c_2}{c_1}})^{-1} = (1 + \sqrt{\frac{A_1(d_0X_{11} - d_1X_{10})^2}{A_2(d_0X_{21} - d_1X_{20})^2}})^{-1}$$

Since K, x_1, x_2, x_0 IT_1, IT_2, are given, the number of devices will be tested under inspection IT_1 and IT_2 are $K_1 = K\pi_1$ (rounded to the nearest integer) and $K_2 = K - K_1$. Therefore, the standard error of reliability is

$$se\big(\hat{R}(t)\big) = \sqrt{\hat{R}(t)} = \frac{R(t)k(\sqrt{A_1(d_0X_{11} - d_1X_{10})^2} + \sqrt{A_2(d_0X_{21} - d_1X_{20})^2})}{\sqrt{KA_1A_2}(X_{10}X_{21} - X_{11}X_{20})}$$

$$= \frac{C}{\sqrt{K}}.$$

With $C = \frac{R(t)k(\sqrt{A_1(d_0X_{11}-d_1X_{10})^2}+\sqrt{A_2(d_0X_{21}-d_1X_{20})^2})}{\sqrt{A_1A_2}(X_{10}X_{21}-X_{11}X_{20})}$.

We can find that standard error of the estimated reliability is in inverse proportion to the square root of the sample size K. Additionally, C is a non-linear function with IT_1, IT_2, x_1 and x_2. The optimization tools could help us to minimize non-linear function, for example **optim** in R. The process that determines the sample allocation in R is,

1. Set stress level x_1 and x_2, inspection time IT_1 and IT_2, the normal condition time x_0, termination time t, and shape parameter k;
2. Compute $(\hat{a}_0, \hat{a}_1)$ that minimizes the C by using an optimization tool;
3. Find $(K_1, K_2, \hat{R}(t))$ with $(\hat{a}_0, \hat{a}_1)$

In the process, $(A_1,\ A_2,\ X_{10},\ X_{11},\ X_{20},\ X_{21})$ could be obtained from $(R(IT_1)$, $R(IT_2)$, IT_1, IT_2, x_1, $x_2)$. C can be derived from $(A_1$, A_2, X_{10}, X_{11}, X_{20}, X_{21}, t, $x_0)$. We could find that the minimum sample size required in the experiment when the standard error of reliability is given,

$$K \geq \left(\frac{C}{se(\hat{R}(t))}\right)^2$$

Consequently, $(K_1,\ K_2)$ can be collected from $(K,\ A_1,\ A_2,\ X_{10},\ X_{11},\ X_{20},\ X_{21})$.

Simulation studies

Fard and Li (2009) conducted an optimal design with Type-I censoring data to estimate the mean lifetime; Alhadeed and Yang (2005) designed optimal times of changing stress level under log-normal cumulative exposure model. In this research, we conduct an analogous simulation study with electro-explosive devices of simple step-stress accelerated life test under Weibull cumulative exposure distribution. The simple step-stress accelerated life test would run to estimate the reliability under the normal operating stress, $x_0 = 25$ with scales parameters $a_0 = 5.5$, $a_1 = -0.05$ and the shape parameter k are discussed in several cases. We will discuss the procedure of optimal design case by case in the following paragraph.

In Table 2 to Table 7, presenting the cases of three situations with failure rate increase ($k > 1$), constant rate ($k = 1$) and decrease ($k < 1$). Under different settings, the optimal design would be examined. Furthermore, the simulation study under each optimal design with 10,000 random experiments and use simulate Binomial function (**rbinom**) in **R** with $1 - R_1$ and $1 - R_2$ failure rate to generate the failure number n_1 and n_2 respectively. Then using **optim** in R to obtain the $\hat{a}_0$ and $\hat{a}_1$, and then using $\hat{a}_0$ and $\hat{a}_1$ to find the optimal sample allocation and estimated reliability. Based on the 10,000 estimated reliability to get the standard error $se(\hat{R}(t))$ of maximum likelihood of reliability under the normal condition and compare the theoretical consequences to the results from the simulation experiments.

Table 2: Optimal design of simple step-stress accelerated life test with k>1, t=60 and corresponding simulated mean reliability and standard error

Setting						SSALT plan		Simulated results		
k	x_1	x_2	IT_1	IT_2	$se(\hat{R}_0(t))$	K_1	K_2	$\hat{\mu}$	$\hat{s}$	R(t)
2	35	55	10	20	0.005	295942	39067	0.4807	0.0064	0.4807
2	35	55	10	20	0.01	73985	9767	0.4805	0.0128	0.4807
2	35	55	10	20	0.05	2959	391	0.4782	0.0637	0.4807
2	35	55	10	20	0.1	740	98	0.4731	0.1200	0.4807
3	35	55	10	20	0.005	1050855	70784	0.5341	0.0058	0.5342
3	35	55	10	20	0.01	262714	17696	0.5339	0.0116	0.5342
3	35	55	10	20	0.05	10508	708	0.5307	0.0577	0.5342
3	35	55	10	20	0.1	2627	177	0.5341	0.1160	0.5342
5	35	55	10	20	0.005	13506041	235051	0.6317	0.0054	0.6318
5	35	55	10	20	0.01	3376510	58763	0.6315	0.0106	0.6318
5	35	55	10	20	0.05	135060	2351	0.6264	0.0535	0.6318
5	35	55	10	20	0.1	33765	588	0.6118	0.1120	0.6318
7	35	55	10	20	0.005	165332166	749340	0.7143	0.0052	0.7143
7	35	55	10	20	0.01	41333041	187335	0.7141	0.0102	0.7143
7	35	55	10	20	0.05	1653322	7493	0.7077	0.0534	0.7143
7	35	55	10	20	0.1	413331	1873	0.6877	0.1173	0.7143

Table 3: Optimal design of simple step-stress accelerated life test with k>1 in different settings and corresponding simulated mean reliability and standard error

Setting							SSALT plan		Simulated results		
k	x_1	x_2	t	IT_1	IT_2	$se(\hat{R}_0(t))$	K_1	K_2	$\hat{\mu}$	$\hat{s}$	R(t)
2	35	55	30	10	20	0.005	55496	7326	0.8325	0.0064	0.8327
2	35	55	30	10	20	0.01	13874	1831	0.8325	0.0013	0.8327
2	35	55	30	10	20	0.05	555	73	0.8193	0.0695	0.8327
2	35	55	30	10	20	0.1	139	18	0.7734	0.1720	0.8327
3	35	55	60	15	30	0.005	328591	35838	0.5342	0.0062	0.5342
3	35	55	60	15	30	0.01	82148	8959	0.5340	0.0125	0.5342
3	35	55	60	15	30	0.05	3286	358	0.5296	0.0623	0.5342
3	35	55	60	15	30	0.1	821	90	0.5131	0.1253	0.5342
5	35	55	60	20	40	0.005	6031806	80695809	0.6292	0.0158	0.6318
5	35	55	60	20	40	0.01	1508951	20173953	0.5891	0.0934	0.6318
5	35	55	60	20	40	0.05	60318	806958	0.4418	0.0879	0.6318
5	35	55	60	20	40	0.1	15080	201739	0.4274	0.0708	0.6318
7	40	60	60	10	20	0.005	41811408	414824	0.7143	0.0042	0.7144
7	40	60	60	10	20	0.01	10452827	103706	0.7142	0.0086	0.7144
7	40	60	60	10	20	0.05	418113	4148	0.7104	0.0434	0.7144
7	40	60	60	10	20	0.1	104528	1037	0.6951	0.0940	0.7144

Table 4: Optimal design of simple step-stress accelerated life test with k=1, t=60

and corresponding simulated mean reliability and standard error

Setting							SSALT plan		Simulated results		
k	x_1	x_2	t	IT_1	IT_2	$se(\hat{R}_0(t))$	K_1	K_2	$\hat{\mu}$	$\hat{s}$	R(t)
1	35	55	60	10	20	0.005	89947	22557	0.4249	0.0071	0.4249
1	35	55	60	10	20	0.01	22487	5639	0.4248	0.0141	0.4249
1	35	55	60	10	20	0.05	899	226	0.4246	0.0713	0.4249
1	35	55	60	10	20	0.1	225	56	0.4234	0.1403	0.4249
1	35	55	60	15	30	0.005	65095	18006	0.4248	0.0073	0.4249
1	35	55	60	15	30	0.01	16274	4501	0.4249	0.0147	0.4249
1	35	55	60	15	30	0.05	651	180	0.4242	0.0738	0.4249
1	35	55	60	15	30	0.1	163	45	0.4194	0.1415	0.4249
1	40	60	60	10	20	0.005	111664	36883	0.4249	0.0056	0.4249
1	40	60	60	10	20	0.01	27916	9221	0.4249	0.0111	0.4249
1	40	60	60	10	20	0.05	1116	369	0.4247	0.0547	0.4249
1	40	60	60	10	20	0.1	279	92	0.4227	0.1064	0.4249
1	40	50	60	10	20	0.005	356063	177112	0.4249	0.0068	0.4249
1	40	50	60	10	20	0.01	89016	44278	0.4249	0.0137	0.4249
1	40	50	60	10	20	0.05	3561	1771	0.4242	0.0669	0.4249
1	40	50	60	10	20	0.1	890	443	0.4224	0.1291	0.4249

Table 5: Optimal design of simple step-stress accelerated life test with k=1 in different settings and corresponding simulated mean reliability and standard error

Setting							SSALT plan		Simulated results		
k	x_1	x_2	t	IT_1	IT_2	$se(\hat{R}_0(t))$	K_1	K_2	$\hat{\mu}$	$\hat{s}$	R(t)
1	35	55	30	10	20	0.005	52920	13271	0.6518	0.0072	0.6519
1	35	55	30	10	20	0.01	13230	3318	0.6518	0.0144	0.6519
1	35	55	30	10	20	0.05	529	133	0.6477	0.0723	0.6519
1	35	55	30	10	20	0.1	132	33	0.6340	0.1457	0.6519
1	40	60	30	10	20	0.005	65697	21700	0.6518	0.0548	0.6519
1	40	60	30	10	20	0.01	16424	5425	0.6518	0.0111	0.6519
1	40	60	30	10	20	0.05	657	217	0.6449	0.0552	0.6519
1	40	60	30	10	20	0.1	164	54	0.6387	0.1100	0.6519
1	40	50	30	10	20	0.005	209488	104204	0.6519	0.0068	0.6519
1	40	50	30	10	20	0.01	52372	26051	0.6517	0.0136	0.6519
1	40	50	30	10	20	0.05	2095	1042	0.6484	0.0671	0.6519
1	40	50	30	10	20	0.1	524	260	0.6388	0.1316	0.6519
1	35	55	60	10	30	0.005	77846	16441	0.4248	0.0067	0.4249
1	35	55	60	10	30	0.01	19462	4110	0.4249	0.0135	0.4249
1	35	55	60	10	30	0.05	779	164	0.4227	0.0664	0.4249
1	35	55	60	10	30	0.1	195	41	0.4149	0.1303	0.4249

Table 6: Optimal design of simple step-stress accelerated life test with k<1, t=60 in different settings and corresponding simulated mean reliability and standard error

Setting						SSALT plan		Simulated results		
k	x_1	x_2	IT_1	IT_2	$se(\hat{R}_0(t))$	K_1	K_2	$\hat{\mu}$	$\hat{s}$	R(t)
0.2	35	55	10	20	0.001	1051578	399369	0.3793	0.0016	0.3793
0.2	35	55	10	20	0.005	42063	15975	0.3793	0.0079	0.3793
0.2	35	55	10	20	0.01	10515	3994	0.3796	0.0157	0.3793
0.2	35	55	10	20	0.05	420	160	0.3977	0.1109	0.3793
0.2	35	55	15	30	0.001	1006602	384640	0.3793	0.0016	0.3793
0.2	35	55	15	30	0.005	40264	15386	0.3796	0.0075	0.3793
0.2	35	55	15	30	0.01	10066	3846	0.3798	0.0159	0.3793
0.2	35	55	15	30	0.05	402	154	0.3981	0.1157	0.3793
0.5	35	55	10	20	0.001	1348200	447165	0.3964	0.0015	0.3965
0.5	35	55	10	20	0.005	53928	17887	0.3965	0.0077	0.3965
0.5	35	55	10	20	0.01	13482	4472	0.3967	0.0153	0.3965
0.5	35	55	10	20	0.05	539	179	0.3986	0.0767	0.3965
0.5	40	50	10	20	0.001	6108686	3726857	0.3965	0.0014	0.3965
0.5	40	50	10	20	0.005	244348	149074	0.3965	0.0070	0.3965
0.5	40	50	10	20	0.01	61087	37268	0.3964	0.0137	0.3965
0.5	40	50	10	20	0.05	2443	1491	0.3970	0.0689	0.3965

Table 7: Optimal design of simple step-stress accelerated life test with k<1 with different settings and corresponding simulated mean reliability and standard error

Setting							SSALT plan		Simulated results		
k	x_1	x_2	t	IT_1	IT_2	$se(\hat{R}_0(t))$	K_1	K_2	$\hat{\mu}$	$\hat{s}$	R(t)
0.2	35	55	30	10	20	0.001	1024286	389004	0.4300	0.0016	0.4300
0.2	35	55	30	10	20	0.005	40972	15560	0.4302	0.0078	0.4300
0.2	35	55	30	10	20	0.01	10243	3890	0.4303	0.0157	0.4300
0.2	35	55	30	10	20	0.05	409	156	0.4442	0.1102	0.4300
0.2	35	55	30	15	30	0.001	980478	374657	0.4300	0.0016	0.4300
0.2	35	55	30	15	30	0.005	39219	14986	0.4301	0.0078	0.4300
0.2	35	55	30	15	30	0.01	9805	3746	0.4300	0.0159	0.4300
0.2	35	55	30	15	30	0.05	392	150	0.4455	0.1097	0.4300
0.5	35	55	60	15	30	0.001	1177537	403181	0.3965	0.0015	0.3965
0.5	35	55	60	15	30	0.005	47102	16127	0.3965	0.0078	0.3965
0.5	35	55	60	15	30	0.01	11775	4032	0.3964	0.0155	0.3965
0.5	35	55	60	15	30	0.05	471	161	0.3991	0.0774	0.3965
0.5	40	50	30	10	20	0.001	5251373	3203817	0.5199	0.0014	0.5199
0.5	40	50	30	10	20	0.005	210055	128153	0.5198	0.0068	0.5199
0.5	40	50	30	10	20	0.01	52514	32038	0.5197	0.0139	0.5199
0.5	40	50	30	10	20	0.05	2101	1281	0.5199	0.0689	0.5199

By and large, it is observed that the mean of the estimated reliability and standard error are very close to the theoretical result. The numerical examples show the shape parameter k increase, the reliability and sample allocation also would increase which means the large hazard rate need a large sample allocation. Meanwhile, shorten the termination time significantly increase the reliability. Changing inspection time IT_1, IT_2, stress level x_1 and x_2 do not provide extra information to collect the reliability information, since reliability only involves termination time and hazard rate.

On the other hand, the inspection time and stress level would have an impact on the optimal sample size. when the shape parameter k is fixed, increasing standard error significantly reduces sample sizes, it means the smaller sample size the more variance because the less information that specimens give. Also, extending both inspection time IT_1 and IT_2 or increasing both stress level x_1 and x_2 significantly reduce sample allocation. Also, the termination time t would give the effect of reliability, prolonging experimental time t would increase its corresponding required sample size. Moreover, more specimens are required with a narrow range of stress levels (increase the stress level x_1 and decrease the stress level x_2). From the numerical examples, in order to effectively collecting data the sample size in the first inspection stage is larger than sample allocation in the second stage.

Limitation

This research aims to make a reference in real application. When having the expected standard error, we could infer the minimum required specimen size; also, the limitation of budget (maximum sample size is fixed) could deduce the standard deviation of the experiment. However, in this project, we assume the hazard rate (shape parameter) is known. The hazard rate in the real application not always be the integer that we considered in this project, even without knowing information about hazard rate. Therefore, known hazard rate is unrealistic.

Also, the inspection time we only considered is equally space in simulation study. We reviewed two cases, IT_1 and IT_2 are 10 and 20, in another case IT_1 and IT_2 are 15 and 30. The test time for devices in first stage and second stage are the same. The unequally space for the inspection time was not considered in this paper, the optimal inspection times may not equal.

Besides, the exact inspection times and stress levels are known but these values we assumed may not the optimal decision for these devices. In real life, many conditions are difficult to controlled so that seldom information was known; sometimes we just know the range of the stress level. Consequently, the settings we deliberated is hard to match up with actual situations.

Conclusion

The simple step-stress accelerated life test for one-shot devices was studied in this thesis. The information matrix was derived and used it to minimize the asymptotic variance. The optimal design is effectively collecting one-shot device testing data that the asymptotic variance of the maximum likelihood estimation of the reliability is minimized. Additionally, process of deciding the optimal sample allocation was analyzed. Furthermore, collected the sample allocation from simulated study at the normal condition with different settings.

Simulation studies with several scenarios show the procedure is absolutely reliable for simple step-stress accelerated life test since the simulated reliability and standard error of reliability are very closed to theoretical results. Also, there are some findings from the simulation studies: 1) When the shape parameter is fixed, the sample allocation would significantly reduce along with increasing the standard error. 2) The reliability and sample size increase when the shape parameter k increases with the same standard error. 3) Prolonging the experiment time t reduce the reliability and more devices are required and vice versa. 4) Changing inspection time IT_1, IT_2 and stress level x_1, x_2 would not give more information about reliability. 5) Decreasing the range of stress level substantially increase the sample allocation. From the simulation study, it is realized that the reliability greatly depends on shape parameter and termination time rather than inspection time and stress level. The experiment get a good effect that sample allocation at least around one-thousand.

In real world, the information about the specimens and the conditions are scarce, the stress levels, inspection time are difficult to control. Based on the restriction of this project, we only considered inspection times are 10 and 20 in first stage and second stage respectively which may not the optimal inspection time for one-shot devices. It is a great practical interest to design more variables in the future research, such as stress level, inspection time and sample allocation which will more correspond to facts.

In this research, we designed the devices tested under only two stages under different stress level and inspection time. It is realized that the determinant of information matrix equals to zero and information matrix has to reduce. If the determinant of information matrix in three or more stages in the simple step-stress accelerated life test exists, the hazard rate does not assume as known parameter which is more practical in the real. It is of great interest to design a simple step-stress accelerated life test under multiple stages and investigate the optimal sample allocation.

The Expectation-Maximization (EM) algorithm is the most powerful technique to effectively derived the maximum likelihood estimation and many scholars have done it in many researches. In our study, there are many censored data. Although Maximum likelihood could find the best model for the data, it does not work well especially for the incomplete data. Expectation Maximization algorithm needs multi-step process to tweak the model to fit the data. But it repeats E-step and M-step until the model is stable which involve much calculus and conditional probability, becomes challenging. In the future research, the Expectation-Maximization algorithm would accomplish with Maximum estimation algorithm so that effectively obtain the maximum likelihood estimation of model parameter. Also, many scholars conducted sensitivity analysis (Ling, 2019) to determine the effect of mis-specification of planning value to the design and examine the robustness of the design. Ling and Balakrishnan (2017) analyzed the mis-specification model for Weibull distribution. The mis-specification parameters commonly exist in real practice, it is worth investigating the mis-specification of the planning values.

Furthermore, decreasing the standard error significantly enlarge the sample size. In practical, it means the cost of experiment will substantial increase. It is of great interest for predict budget so that researcher could make a better decision with a limited funds.

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