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Title: Exploring Ways of Proving the Missionaries and Cannibals Problem and the Teaching Implication

Outline

I. Introduction

My honour project proposal begins in investigating river-crossing game. During our honour project lectures, I found that nearly all my friends in the math major have played this game before, so I have the intention is it is a popular game which is worth investigating into. Not only me having the impression of river-crossing game is popular, but also Ascher (1990) mentioned that it is a cross-cultural game.

In the below essay, I am going to divide it into a few parts, including the rationale, literature review, using the Dijkstra's algorithm and Fraley et al. (1966) method to solve the problem, in addition to the teaching implication plus research results.

II. Rationale of Selection of the topic

According to Ascher (1990), the history of river-crossing problems could be dated back to 1000 years ago. There were letters found that at that time, people from the whole world, including the West and the East, like to talk about how to solve the river-crossing problem. It is because they would like to solve real-life problems. The ships are still famous today, so I think it is never too old to talk about solving river-crossing problems.

The math problem has also been generated into computer games. I like mathematical games too, and the below one is the picture captured from the website Little Campus about the online river-crossing game that I would like to play during childhood. Little campus is a website developed by the Hong Kong Education City which aims to provide more resources for learning, such as mathematical and logical games. We could see that math game is promoted by the government as well and this thesis could help to draw insights on how to teach students to play skillfully, not just trial and error and increase their sense of math.

*1: photo from Little Campus of the river-crossing game (FungCC, 2016)(the photo is captured from YouTube video instead of the official website of Little Campus, as Little Campus has removed the game already.)

There is a journalist Bragg (2012) in addition to Bakker et al. (2015) suggested that Math game could help to increase on-task behavior. They would pay more attention, so I think that my research could drive me to think of how to use mathematical concepts to solve and give implications on how to teach students to think about logical games, not just using logic, but appreciate how we could use math solution to solve intrinsically.

In addition, I had registered a general education class in the Education University of Hong Kong 2 years ago, which is called Mathematics Make Life Simpler. In the lesson, I learned to solve graphs using Dijkstra’s algorithm. I found that interesting to investigate.

The research questions of my honour project are listed below:

1. How could we use Dijkstra’s algorithm to solve the missionaries and cannibals’ problem?
2. If we alter the missionaries and cannibals’ problem that the cannibals are heavier than the missionaries so that we need more effort to paddle them, in which we indicate this by giving weights to edges which represent the effort to paddle, how could Dijkstra’s algorithm be applied?
3. How could Fraley et al. (1966) method be used in the river-crossing problem that the boat could only carry odd number of passengers?
4. 林炳炎(2010) has said that on his blog that there are at least 7 crossings for the missionaries and cannibals' river crossing problem if the boat has the capacity of odd numbers of people bigger than 3, and the boat capacity is smaller than the number of missionaries or cannibals by 1. How could we use examples to prove that if there are any discrepancies?
5. River-crossing game is a mathematics game. How could the playing of math game help students’ in development? How could we

improve that?

III. How to Play the missionaries and Cannibals Problem?

Please refer to appendix A for the game rules.

IV. Dijkstra's Algorithm

A. Literature Review

Please refer to appendix B.

B. Applying Dijkstra's Algorithm into the Missionaries and Cannibals Problem

a. Abbreviations of the Math Problem

To start, let us first define some abbreviations for the names. Let M and C denote missionaries and cannibals. We do not assign any numbers to the respective missionaries or cannibals as it makes no difference among them.

b. Listing out all the Possible States

Let's see how many states there at most could have. The left-hand side of the bar represents the starting side of river-crossing and the right-hand side of the bar the destination. We also assign alphabets to address each state. There are 10 states in total.

	Left	Right	Name of Vertex
(start)	MMMCCC		A
(goal)		MMMCCC	B
	MMMCC	C	K
	C	MMMCC	D
	MMMC	CC	E
	CC	MMMC	F
	MMM	CCC	G
	CCC	MMM	H
	MMCC	MC	I
	MC	MMCC	J

Some cases seem alright to go, but it is not a possible state.

For example,

MMC	MCC

It appears as on the left-hand side of the river it is okay that the number of missionaries outweighs the number of cannibals. However, on the right-hand side of the river, the number of missionaries is nerve-weakening, in which they would be eaten by the cannibals as number of C exceeds the number of M.

C. Classifying the States into 3 Cases

We could observe from the above 10 states that the number of missionaries and cannibals must be the same on each side of the river in order to maintain harmony if there exists both missionaries and cannibals on the same side. The cases are I and J.

There are other 2 cases of the states that we could categorize. First, for A and B, they are the states that all missionaries and cannibals are staying together in the start and goal state.

Second, the states C, D, E, F,G and H is how we are just moving the cannibals to the other side of the river, giving the missionaries to be together all the time.

The edges are as follows. The states in bracket represent meaningless states.

A→K, E, I

B→(meaningless, we won't go back once complete the journey)

K→(A),E, G, I (A is meaningless, as it goes back from the start, which is a loop on the graph)

D→B, F, H, J

E→(A), K, G, J

F→ B, D, H, I

G→K, E

H→D, F

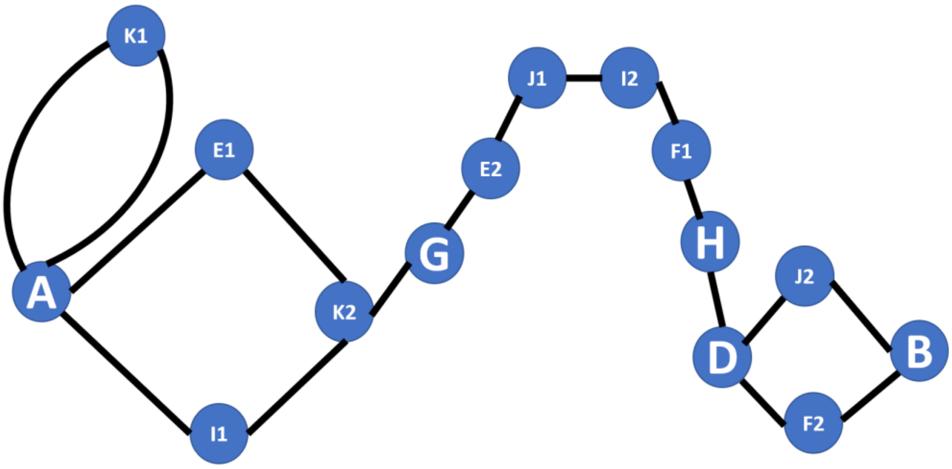
I→(A), K, F, J

J→B, D, I

The states that are repeated would be denoted differently with numbers beside the alphabet on the graph, as some states would appear more than one time in the transition process. For example, K has repeated two times as for one time it is a loop that one cannibal goes to the destination side of the river and then goes back to the starting point, which would be denoted as K1; K2 is denoted that we sends one cannibal to paddle the boat back to the original side of the river in order to carry out the next journey.

D. Performing Dijkstra’s Algorithm

Now let’s perform Dijkstra’s algorithm to find out the shortest distance based on the following graph drawn from the above list, in which shortest distance means the sum of the weights and shortest path means the sequence of the vertices we found. In our case, they are the same now as the weight of each edge is 1.



*2: the graph for missionaries and cannibals problem

So, in the above graph, all edges carry the weight of 1, in which we count the edges as the steps to reach from A to B. Each edge would only be walked once. Our approach is to use Dijkstra's algorithm to find the optimum distance to go with fewer steps.

Let's create the table of Dijkstra's algorithm Shiu and Ling's (2013) model.

The 1st Step: Classify all the vertices into I(visited), II (current) and III(unvisited).

The 2nd Step: Put the starting vertex into II (current), and all the other vertices are put in III(unvisited).

The 3rd Step: Determine the distance to travel to other vertices from the starting point and write down the corresponding tendency distance (abbreviated as T.D.).

The 4th Step: Move the II (current) vertex to I(visited).

The 5th Step: Choose the vertex of the smallest weight to move from III (unvisited) to II (current) to become the new current vertex. If there is more than one vertex with the smallest weight, we decide according to alphabetical order.

The 6th Step: Repeat until all vertices are visited. Note that the ∞ sign would be used to represent the unknown weight of the edge.

Let us start by the following.

	I	II	III													
Vertex		A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.		0	1	1	1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

1. Move A to I(visited) and find another point to visit. By our practice, when the distances are all the same, we will visit the vertex with the smaller alphabetical order first, so we choose K1.

	I	II	III												
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	1	1	1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

2. Repeat the steps.

	I		II	III											
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	1	1	1	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I			II	III										
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	1	1	1	2	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I				II	III									
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	1	1	1	2	3	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I					II	III								
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	1	1	1	2	3	4	∞	∞	∞	∞	∞	∞	∞	∞

	I						II	III							
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	1	1	1	2	3	4	5	∞	∞	∞	∞	∞	∞	∞

	I							II	III						
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	1	1	1	2	3	4	5	6	∞	∞	∞	∞	∞	∞

	I								II	III(unvisited)					
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
	0	1	1	1	2	3	4	5	6	7	∞	∞	∞	∞	∞

	I									II	III				
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
	0	1	1	1	2	3	4	5	6	7	8	∞	∞	∞	∞

	I										II	III		
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	F2	B
T.D.	0	1	1	1	2	3	4	5	6	7	8	9	∞	∞

	I											II	III		
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	F2	J2	B

T.D.	0	1	1	1	2	3	4	5	6	7	8	9	10	10	∞

	I											II	III	
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	F2	J2	B
T.D.	0	1	1	1	2	3	4	5	6	7	8	9	10	10	11

	I													II	III
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	F2	J2	B
T.D.	0	1	1	1	2	3	4	5	6	7	8	9	10	10	11

	I														II	III
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	F2	J2	B	
T.D.	0	1	1	1	2	3	4	5	6	7	8	9	10	10	11	

	I(visited)															II	III
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	F2	J2	B		
T.D.	0	1	1	1	2	3	4	5	6	7	8	9	10	10	11		

So now all the vertices are visited. The weight of the edges to B is 11. And there are four ways to go from A to B, which are

A → E1 → K2 → G → E2 → J → I2 → F1 → H → D → F2 → B,

A → I1 → K2 → G → E2 → J → I2 → F1 → H → D → F2 → B,

A → E1 → K2 → G → E2 → J → I2 → F1 → H → D → J2 → B,

Or A → I1 → K2 → G → E2 → J → I2 → F1 → H → D → J2 → B

In 11 steps we could let all the cannibals and missionaries go to their destination side of river.

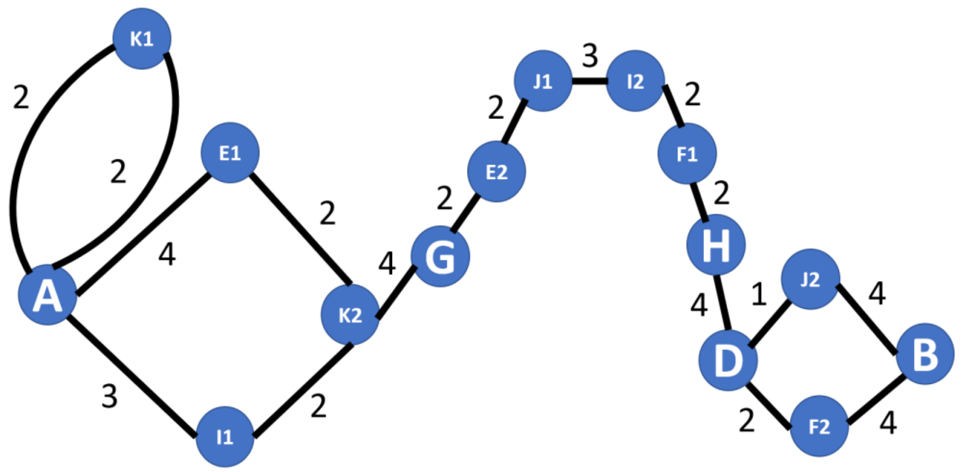
Another thing to note is that we do not pass through the vertex K1, so it is not a Hamiltonian path.

E. Altering the Problem into a More Interesting Problem

Let us change the problem a little bit to turn it into a more interesting and more complicated question to solve. Let us suggest that cannibals are heavier than missionaries as they eat more meat. We put the effort of paddling the boat with each missionary as 1 and each cannibal as 2. Now we apply Dijkstra's algorithm again.

For instance, from A to E1 we have the effort of 4 because we are carrying 2 cannibals to the destination side of the river; while from A to I1 we have the effort of 3 because we are only carrying 1 missionary and 1 cannibal.

We have the following graph, with weighted edges.



*3: the graph for altered missionaries and cannibals problem

The same steps are repeated and please refer to appendix C for the process, while the final solution would be shown here.

	I(visited)													II	III
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
	0	2	3	4	5	9	11	13	16	18	20	24	25	26	29

	I(visited)														II	III
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B	
	0	2	3	4	5	9	11	13	16	18	20	24	25	26	29	

	I(visited)															II	III
Ver-	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B		
tex	0	2	3	4	5	9	11	13	16	18	20	24	25	26	29		

From the above result, we could find out the shortest distance from A to B is 29, in 11 steps, so the path is A → I1 → K2 → G → E2 → J → I2 → F1 → H → D → J2 → B. In this way, we could save the manpower for paddling missionaries and cannibals to the other side of the river.

In this case, we consider the shortest distance and find out the path to save the effort of paddling and manpower. As the weights of different edges are different, we need to compare the distance cost first, then find out the path. In other words, the shortest path we found may not have the shortest distance.

V. **Fraley et al. (1966) Method**

Here comes another way to solve the missionaries and cannibals problem. We are going to review the graphical method suggested by Fraley et al. (1966) on finding general solutions. As he had only worked out the representation of the condition of the boat capacity of even number of people, my paper is going to work out the condition of the boat capacity of odd number of people crossing different states, where the boat capacity is bigger than three. And as suggested previously, the 'states' mean the collection of varying combination of cannibals and missionaries on the 2 sides of the river.

After that, we would challenge the general method found by 林炳炎 (2010) and explore more conditions of river crossing. Finally, for the sake of improving problem-solving methods, we would add on Fraley et al (1966)'s method.

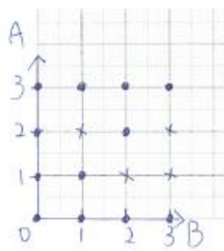
A. Literature Review

a. Putting the possible and impossible states into a coordinate system

Fraley et al. (1966) suggested how we could show the transition of missionaries and cannibals using the coordinate system. As indicated by them, there must be somebody paddling the ship back and forth of the river. They used straight lines for the travel towards the destination, and dotted lines for the travel back from the destination to the starting side of the river.

First, we need a general graph for all the possible states on both sides of the river. In the graph (refer to *4), A represents missionaries and B represents cannibals. The dots "." represent the possible states of the number of cannibals and missionaries. The crosses "x" represent the impossible conditions, in which the cannibals will eat the missionaries as the number of cannibals outweigh that of missionaries.

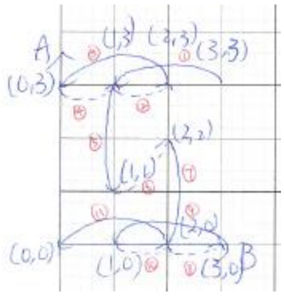
For example, (2,2) means 2 cannibals and 2 missionaries are on one side of the river. It is a possible case as the number of cannibals is the same with the number of missionaries. We could also think of the other side of the river the number would be (1,1), with 1 cannibal and 1 missionary, and that would be a peaceful condition too. While (2,1) would be an impossible situation that though it seems okay that on this side of the river we have 2 missionaries and 1 cannibal, in which the number of missionaries outweighs that of cannibals. However, on the other side of the river, there would be 1 missionary and 2 cannibals, which is an impossible situation. Please refer to the following graph.



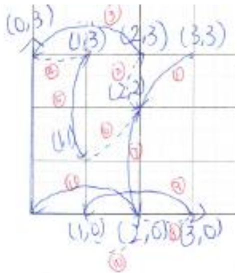
*4: the possible and impossible states

(missionaries is represented as A on y-axis and cannibals as B on the x-axis)

b. **Linking the dots to form a trip**



*5: One of the possible ways to travel (B=2, m=c=B+1=3)



*6: One of the possible ways to travel (B=2, m=c=B+1=3)

Graphs 5 and 6 show some of the ways to travel, in which they are based on graph 2 and we draw arrows to represent the forward steps from original to destination side and dotted arrows to represent the returning steps. For illustration, we use graph 4 to explain. The steps are listed below.

1. First, we let (3,3) be the starting point.
2. We have (2,2) where 1 missionary and 1 cannibal crossed.
3. 1 missionary goes back so we have (2,3), which is represented by the dotted line.
4. Two cannibals go, and we have (0,3).

5. Let "→" means forward, "--->" means backward. Repeat the steps.

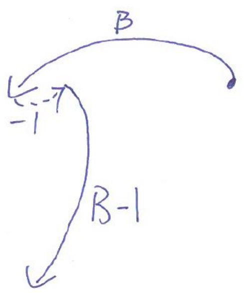
The path goes as

(3,3) → (2,2) ---> (2,3) → (0,3) ---> (1,3) → (1,1) ---> (2,2) → (2,0) ---> (3,0) → (1,0) ---> (2,0) → (0,0)

Using the above method, Fraley et al (1966) had found out some generalized patterns of river-crossing problem. If the boat could hold four or greater number of even people, there are two methods. It is illustrated below (refer to *7 and *8). The writers come out with the conclusion that generally, the boat capacity is B, in which B is an even number and B ≥ 4, we have two general methods.

The first method is listed in the steps below. 3 steps are needed to send B – 1 missionaries and cannibals out respectively.

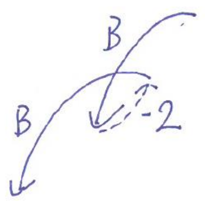
1. Send B cannibals to the other side of the river.
2. Return 1 cannibal.
3. Send B – 1 missionaries out.



*7: The first method(General graph)

The second method is again listed below. It also includes 3 steps.

1. Transport $\frac{B}{2}$ numbers of cannibals and $\frac{B}{2}$ of missionaries to the other side of the river.
2. Returning 1 cannibal and 1 missionary, so in total returning 2 people.
3. Again transport $\frac{B}{2}$ numbers of cannibals and $\frac{B}{2}$ of missionaries



*8: The second method(General graph)

B. **<u>The Conditions</u>**

Using Fraley et al (1966) method), we are going to work out with the boat capacity of odd number larger than or equal to 3(i.e. B ≥ 3), while the number of cannibals and missionaries are B + 1 respectively. Moreover, missionaries would be abbreviated as m, and cannibals as c in representation.

So, we have the following:

1. Boat capacity = B

2. Number of m: B + 1

3. Number of c: B + 1

Also, the starting side of the river would be called as the original side and the side that we are going to would be the destination side.

Our representation would first include all the diagrams of all the different steps of river crossing. Then, we would represent the general transition of states using Fraley et al. ’s (1966) method.

The right column would be the starting side of the river, so the missionaries and cannibals should cross to the left-hand side of the river, which is the destination side. The number before “m” or “n” means the number, for example, 10m means we have 10 missionaries on this side of the river. The solid arrows mean the action of crossing the river from the starting side to the destination side of the river. Whereas, dotted arrows mean the vice versa. “S” means step, so “S1” means step 1.

The following representation and diagram are adapted and adopted from the references of the Missionaries and Cannibals Problem(n.d.), which has used dots and triangles to represent the river crossing. Pressman and Singmaster (1989) have used different letters to represent the wives and husbands of the Jealous husband problem. I do not use the transition graphs of Fraley et al. (1966) as shown in *5 and *6, as they are uneasy to read to find out the steps. To adopt into my own created method, I used numbers before the letters to represent the missionaries and cannibals for easy reference. It is because I found that the readers would feel tedious to count the number of dots or other keys when we got a larger number of missionaries or cannibals.

C. **<u>Listing out Some of the Cases</u>**

In the following, we are going to use 5 cases to analyze the general solutions. We would have 3 methods to transfer them to the destination side of the river. Method 1(M1) is to transfer B cannibals to the destination first in four steps; then we transfer B number of missionaries in the 5^{th} step. Method 2(M2) is that we first transfer B cannibals to the destination first, and then back 1 cannibal. In the 3^{rd} step, we transfer B-1 missionaries to the destination. Method 3(M3) is we transfer $\frac{B-1}{2}$ number of cannibals and missionaries respectively to the destination in the 1^{st} step, then we transfer back 1 missionary and 1 cannibal in the 2^{nd} step. We transfer all the missionaries to the destination in the 3^{rd} step.

While using all the three methods, in the next sessions, we would consider which methods are better based on the definition of good method as utilizing the fewest steps to complete the mission.

a. Case 1 (B = 3, m = c = B + 1 = 4)

<u>M1(A)</u>

(Start)			
S1		4m 4c	To 3c
S2	3c	4m 1c	Back 1c
S3	2c	4m 2c	To 2c
S4	4c	4m	Back 1c
S5	3c	4m 1c	To 3m
S6	3m 3c	1m 1c	Back 1c + 1m
S7	2m 2c	2m 2c	To 1c + 2m
S8	4m 3c	1c	Back 1m
S9	3m 3c	1m 1c	To 1c + 1m
(Goal)	4m 4c		

S1: To 3 cannibals. /To B cannibals.

S2: Back 1 cannibal

S3: To 2 cannibals

S4: Back 1 cannibal

S5: To 3 missionaries. /To B missionaries.

S6: Back 1 cannibal and 1 missionary

S7: To 1 cannibal and 2 missionaries

S8: Back 1 missionary

S9: To 1 cannibal and 1 missionary. / To all the remaining people.

However, steps 6 to 9 could vary but still, involve 9 steps. They are illustrated below.

In step 7, we must transfer all the missionaries to the other side of the river, as if we do not do so, there would be an imbalance on the starting side. For instance, if we transport 1 missionary and 1 cannibal, they would be transported back to maintain balance in the next step, so nothing has actually been done. The critical consideration in step 7 is that we transport all the remaining missionaries to the destination side of the river, but whether we would transport any cannibals. We could see that even we don't further transport any cannibals, they could all be transported in step 9. One of the possible methods is that we transfer 2 missionaries only in step 7, then we could transport one cannibal back in step 8, so that totally 3 cannibals could be transport forward in step 9, which is illustrated in M1(B).

M1(B):

S7	2m 2c	2m 2c	To 2 m
S8	4m 2c	2c	Back 1 c
S9	4m 1c	3c	To 3 c
(Goal)	4m 4c		

M2(A):

(Start) S1		4m 4c	To 3c
S2	3c	4m 1c	Back 1c
S3	2c	4m 2c	To 2m
S4	2m 2c	2m 2c	Back 1c + 1m
S5	1m 1c	3m 3c	To 3m
S6	4m 1c	3c	Back 1c
S7	4m	4c	To 3c
S8	4m 3c	1c	Back 1m
S9	3m 3c	1m 1c	To 1c + 1m
(Goal)	4m 4c		

S1: To B cannibals

S2: Back 1 cannibal

S3: To B – 1 missionaries

S4: Back 1 cannibal and 1 missionary

S5: To B missionaries

S6: Back 1 cannibal

S7: To B cannibals

S8: Back 1 missionary

S9: To 1 cannibal and 1 missionary

Again, step 8 could be varied as below.

M2(B)

S8	4m 3c	1c	Back 1c
S9	3m 2c	2c	To 2c
(Goal)	4m 4c		

Method 3 could not be used here as we would be transferring 1 missionary and 1 cannibal across the river in step 1, then back 1 missionary or 1 missionary and 1 cannibal in step 2, which is meaningless that a balance could not be achieved.

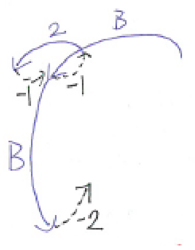
So, for case 1, both M1 and M2 are the best methods.

Cases 2 to 5 would be put into appendix D.

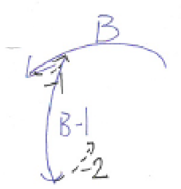
D. Patterns found

a. Comparing the Three Methods

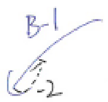
3 methods are used to solve the river-crossing of missionaries and cannibals, where the graphs are as follows.



M1: Crossing B cannibals and B missionaries in 6 steps



M2: Crossing B – 2 cannibals and B – 2 missionaries in 4 steps



M3: Crossing $\frac{(B-1)}{2}-1$ cannibals and $\frac{(B-1)}{2}-1$ missionaries in 2 steps

b. Method 1

Method 1 is an excellent method when the boat capacity is small. This is because step 3 in every case using of method 1, we are transferring 2 cannibals to the destination side of the river. It would waste the time of crossing when the boat capacity is larger than 3. In cases 1 and 2, it remains a suitable method, but in case 3 we could see that it costs 7 steps, but the other methods only cost 5 steps, in particular in case 3, we waste the capacity of B – 2 =7 people to cross the river in the 3rd step. Further calculations are shown below.

The number of missionaries left on the original side of the river by using method 1 after 6 steps:

= original number of missionaries – (missionaries carried to the destination side of the river in the first 5 steps) + (missionaries sent back to the original side of the river in step 6)

= (B + 1) – B + 1

= 2 missionaries

Number of cannibals left on the original side of the river by using method 1 after 6 steps:

= original number of cannibals – (people carried to the destination side of the river in step 1 and 3) + (people sent back to the original side of the river in step 2,4 and 6)

= (B + 1) – B + 1 – 2 + 1 + 1

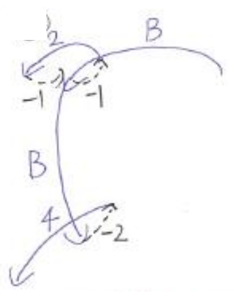
= 2 cannibals

Total number of missionaries and cannibals left on the original side of the river by using method 1 after 6 steps:

= 2 + 2

= 4 people

After step 6, by simple calculation and observation, we could see that there would always be 2 missionaries and 2 cannibals left on the original side of the river. For case one, as the boat capacity is only 3, so we need 3 more steps to transfer the remaining missionaries and cannibals. While for case two and on, we have known that the boat capacity must be greater than 3, so all the remaining 4 cannibals and missionaries could be transferred in one step. Therefore, method 1 would always yield 7 steps to solve for all cases that the boat capacity is an odd number bigger than 3. You could find the following graph for illustration



* M1 ($B > 3$ and $B = 2k + 1$, where $k \in \mathbb{N}$)

c. **Method 2**

For method 2, it is by far the best method. The steps are as follows.

S1: To B cannibals

S2: Back 1 cannibal

S3: To B – 1 missionaries

S4: Back 1 cannibal and 1 missionary

In four steps, we would transfer B – 2 cannibals and B – 2 missionaries to the destination side of the river. Comparing this method to method 1, in which only B cannibals are transferred, we could observe that when B is larger, method 2 would be more efficient to transfer people than method 1.

Total number of people transferred using method 1 in the first four steps:

Step 1	Step 2	Step 3	Step 4
+ B	– 1	+ 2	– 1

= B people

Total number of people transferred using method 2:

Step 1	Step 2	Step 3	Step 4
+ B	– 1	+ (B – 1)	– 2

= 2B – 4 people

The number of missionaries left on the original side using method 2 after the first four steps,

=original number – the number of missionaries transported to the destination side in step 3 – the number of missionaries transported back to the original side in step 4

$= (B + 1) – [(B – 1) – 1]$

$= 3$

The number of cannibals left on the original side using method 2 after the first four steps,

$= (B + 1) – [(2B – 4) – (B – 2)]$

$= 3$

The number of people left for transportation on the original side after the first 4 steps,

= 6 people

So, there would be 6 people left after the first 4 steps for all the cases.

We could also derive that when the boat capacity is larger than 6, the river crossing could be completed using this method in 5 steps. If the boat capacity is smaller than 6 but bigger than 3, totally there would be 7 steps. If the boat capacity is 3, 9 steps are needed. It could also be proved by mathematical induction.

When $B = 3$, the number of people left for transportation on the original side after the first 4 steps,

$= 2B + 2 - (2B - 4)$

$= 2 \times 3 + 2 - (2 \times 3 - 4)$

$= 6$ people

When $B = 5$, the number of people left for transportation on the original side after the first 4 steps,

$= 2B + 2 - (2B - 4)$

$= 2 \times 5 + 2 - (2 \times 5 - 4)$

$= 6$ people

Assume it is true for $B=k$.

Put $B = k$, the number of people left for transportation on the original side after the first 4 steps,

$= 2B + 2 - (2B - 4)$

$= 2 \times k + 2 - (2 \times k - 4)$

$= 6$ people

When $B = k + 1$, the number of people left for transportation on the original side after the first 4 steps,

$= 2B + 2 - (2B - 4)$

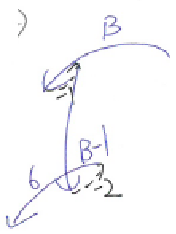
$= 2 \times (k+1) + 2 - [2 \times (k+1) - 4]$

= 6 people

$\therefore$ P(k + 1) is also true.

$\therefore$ P(B) is true for all positive integers.

Therefore, when the boat capacity is an integer greater than or equal to 6, it would only totally include 5 steps to transfer all the people to the destination side, which could be applied to both the even and odd cases of boat capacity.



* M2 ($B \geq 5$ and $B = 2k + 1$, where $k \in \mathbb{N}$)

d. **Method 3**

For method 3, in the 1st step we transfer the same number of cannibals and missionaries to the other side of the river to maintain a balance of them. As our boat could only carry odd number of people in this step, in order to maintain a balance, we carry $\frac{(B-1)}{2}$ of missionaries and cannibals to the other side of the river. In the 2nd step, we transport back 1 missionary and 1 cannibal to maintain a balance as well. In the 3rd step, we transfer all the missionaries first as our goal, so all the missionaries could gather all together at the destination side.

This method has an advantage because all the missionaries are staying together; there is no way that the cannibals' number would outweigh them. Therefore, whatever steps we are performing later, the backward action could only include transferring 1 cannibal back to get back the boat for forwarding travelling. In other words, we do not need to transfer one missionary and one cannibal together at one step to maintain the balance, so the transportation cost reduced. This method was thought because Fraley et al (1966) made use of this diagonal approach, in which in every forward step he would transfer $\frac{B}{2}$ cannibals and $\frac{B}{2}$ missionaries, but in our condition, it is not an efficient method to do in every step. We got odd number of boat capacity, so we could not make use of the boat capacity to the full extent.

The steps of my method 2 are recaptured as follows.

S1: To $\frac{(B-1)}{2}$ cannibals and $\frac{(B-1)}{2}$ missionaries

S2: Back 1 cannibal and 1 missionary

S3: To all missionaries, and could choose to transport some of the cannibals

S4: Back 1 cannibal

S5: To all the remaining cannibals

There may be more steps involved if the remaining cannibals could not be transferred in one step.

Method 3 is a great method except for the condition of case 1 that the boat capacity is only 3. In this case it would be useless that when step one we transfer 1 cannibal and 1 missionary to the other side of the river, in step 2 we transfer 1 cannibal and 1 missionary back. It would be a very efficient method when B gets larger, which is illustrated as follows.

The number of missionaries and cannibals carried respectively to the destination side of the river in the first two steps:

$= \frac{(B-1)}{2} - 1$

$= \frac{(B-3)}{2}$

The number of missionaries and cannibals left respectively on the original side of the river after the first two steps:

$= (B + 1) - \frac{(B-3)}{2}$

$= \frac{(B+5)}{2}$

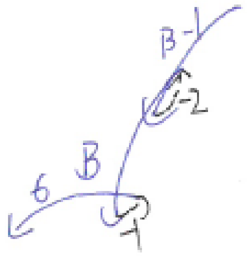
The total number of missionaries and cannibals left on the original side of the river after the first 2 steps of method 3:

$= 2 \times \frac{(B+5)}{2}$

$= (B+5)$

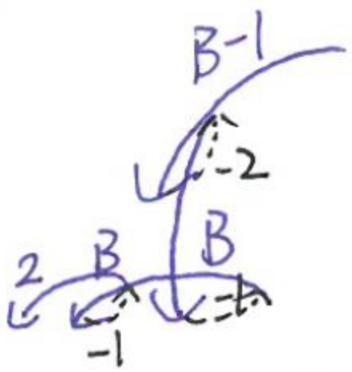
For the cases of B larger than 5, the number of missionaries left on the original side of the river must be smaller than the boat capacity. However, the total number $(B+5)$ is bigger than the boat capacity. As B gets larger, in step 3, we could carry some of the cannibals, not only missionaries. When B gets larger, the ‘5’ added to the B is comparatively small, and when the sum is divided by 2, it would be only larger than $\frac{B}{2}$ by a bit, but it should be always greater than $\frac{B}{2}$. As a result, the people left on the original side of the river could not be carried in 1 step after the first 2 steps. In step 3, we would carry all the missionaries and some of the cannibals to the destination side of the river. While in step 4, we would send 1 cannibal back. In the final step, all the remaining cannibals could be sent to the destination. In total, we use 5 steps to do. Nonetheless, we need not transport some cannibals with the missionaries in the 3rd step, with the varied approaches as seen below.

1. **Approach 1**



* Approach 1: B $\geq$ 6

In approach 1, we transport B number of missionaries and cannibals all together in the 3rd step. Then, we transport 1 cannibal back in the 4th step. In the final step, we would transport 6 cannibals to the destination side of the river if the boat capacity is larger than or equal to 6.

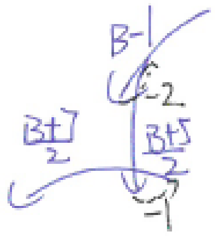


*Approach 1: B=5

For the case of B = 5, as the boat capacity is smaller than 6, so we need two more steps to transport the remaining cannibals. We could choose to transport varied numbers of cannibals in the step 6 and 7, as you could see in the previous examples.

2. **Approach 2**

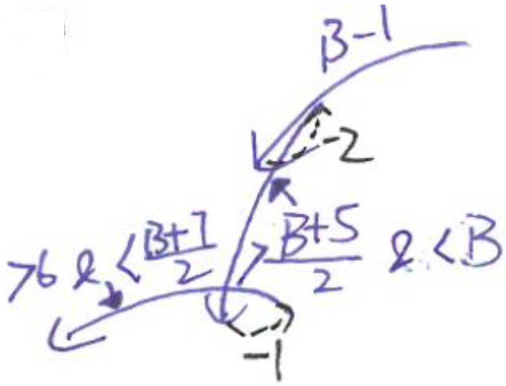
In this approach, we just transport $\frac{(B+5)}{2}$ number of missionaries to the destination side of the river in step 3. While in step 4, we transport 1 cannibal back. In step 5, we transport $\frac{(B+7)}{2}$ of cannibals.



*Approach 2: B ≥ 6

3. **Approach 3**

Approach 3 is an approach that we transport some cannibals with the missionaries in step 3. The diagram is shown below. This approach is that of the combination of method 1 and method 2.



*Approach 3: B > 3

E. **The Person Who is Sent to Paddle the Boat back**

There is another conclusion we could draw on the backward pattern of the river crossing problem. In order to maintain a balance, we would send back 1 missionary and 1 cannibal to get back from the destination side to the original side when not all missionaries have arrived the destination side yet. However, when all the missionaries have arrived on the destination side, we would send 1 cannibal as the one who paddles the boat back. Sometimes sending one missionary back is okay too. It depends on the number of missionaries and cannibals on both sides in order to strike a balance.

For the cases of the boat capacity becoming bigger, we could observe that the transport of missionaries and cannibals become more efficient, that we could transport a large number of passengers one time to the destination side of the river, while backing that we only send back 1 cannibal, or sometimes 1 missionary and 1 cannibal together to strike a balance. It is similar to the situation of Hong Kong that we have buses for public transport. We have only one driver, but we transport each driver with a large number of passengers, where our buses are of double-deck. It could improve efficiency and be environmental-friendly.

F. <u>Summing up the Three Methods</u>

In our case studies and discussion, we found out 3 methods.

Method 2 would be the best method that we try to get B cannibals to cross the river first, then return 1 cannibal and let B – 1 missionaries to cross the river. Then, we back 1 cannibal and 1 missionary. After that, we try to transfer all the missionaries and some cannibals. Finally, we transfer the remaining cannibals.

Method 1 is not quite a good method that in the first 4 steps, we transfer B cannibals. The steps 1 and 2 are the same with method 2, but in step 3 we transfer 2 cannibals to the other side of the river such that in the next step, we transfer 1 cannibal back and in step 5 we could transfer the maximum boat capacity B missionaries to compensate that. However, whenever large is B, step 3 could only transfer 2 cannibals, and there is no choice to add more cannibals to the boat in this step, so that it would be a waste.

Method 3 has a similar concept with method 1 that we would like to transfer the missionaries as our priority. However, we could not transfer the missionaries in the first step, as it would cost imbalance due to the fact that not all missionaries could be transferred in the first step. Therefore, we try to transfer ((B-1))/2 cannibals and missionaries, respectively to the other side of the river first. In the second step, we transfer 1 cannibal and 1 missionary back. In the third step, we transfer all missionaries to the other side of the river. Finally, we try to back 1 cannibal and transfer all the cannibals in the following steps.

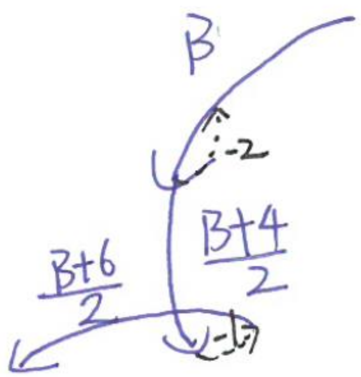
G. Comparison of the Methods Suggested by Fraley et al. (1966)

The method two above is the same as the method suggested by Fraley et al. (1966). In the first step, we transfer B cannibals to the other side of the river. While in the second step, we transfer back one cannibal. In the third step, we transfer B – 1 missionaries. In the fourth step, we send back one missionary and one cannibal. It could be concluded that this method is suitable for both odd and even number of boat capacity.

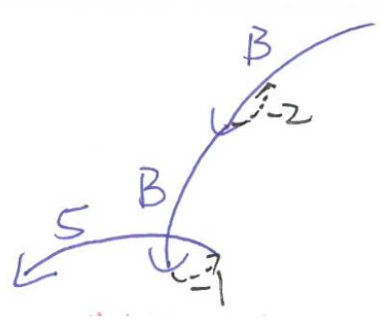
For method 3, it is a similar method when compared to the second method suggested by Fraley et al. (1966). We make use of the diagonal of the coordinate. The writer pointed out that when two points are apart, a straight line would be the shortest distance to go from one point to another. While on the same coordinate plane, we have the same number of missionaries and cannibals, so the shortest distance for travelling on the graph should be diagonal.

H. Enriching Fraley et al (1966)'s Method

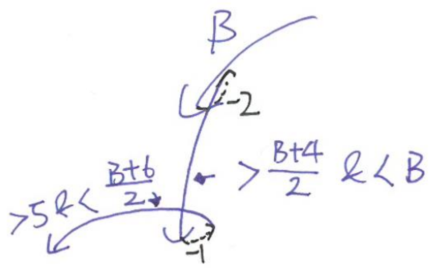
Nonetheless, for my odd number boat capacity situation, it is not good to use the "diagonal method" suggested by Fraley et al. (1966). It is because Fraley et al. (1966) could make use of the full boat capacity to transport $\frac{B}{2}$ cannibals and missionaries respectively, in total we transfer B number of passengers. Whereas, in my odd number boat capacity cases, I could only transport B – 1 people in total in one trip, which is not efficient. Therefore, I tried to be more productive by having the method 3 approaches 1 to 3. It is also excellent to be used in the even number boat capacity condition, where the diagrams are shown below. For the cases of B > 5, we could use the following methods. With B < 5, we could use 2 more steps to transfer all the remaining cannibals to the destination side of the river.



*Method 3 Approach 1 for odd number boat capacity: B > 5 and even number of passengers



*Method 3 Approach 2 for odd number boat capacity: B > 5 and even number of passengers



*Method 3 Approach 3 for odd number boat capacity: B > 5 and even number of passengers

The examples of cases are left for readers to do, while the detailed calculations of the graphs of the three approaches are as follows.

The number of missionaries left on the original side of the river after the first two steps:

$(B + 1) - \frac{B}{2} + 1$

$= \frac{2B+2-B+2}{2}$

$= \frac{B+4}{2}$

Therefore, for approach 1, step 3, as we are only carrying all the missionaries to the destination side of the river, so there would be $\frac{B+4}{2}$ missionaries to cross.

For approach 2 step 3, we just add the maximum number of cannibals to cross with the missionaries, so in total there would be B number of passengers cross together.

For approach 3 step 3, we add a certain number of cannibals to go with the missionaries.

The number of cannibals left for step 5 to carry in approach 1:

$(B + 1) - \frac{B}{2} + 1 + 1$

$= \frac{2B+2-B+2+2}{2}$

$= \frac{B+6}{2}$

The number of cannibals left for step 5 to carry in approach 2:

$(B + 1) - \frac{B}{2} + 1 - (B - \frac{B+4}{2}) + 1$

$= \frac{2B+2-B+2-2B+B+4+2}{2}$

$= \frac{10}{2}$

$= 5$

So, using approach 2, we would always carry 5 cannibals in step 5 when the boat capacity is greater than or equal to 5. It is a similar case for the odd number cases I did in the previous sections.

While in approach 3, we would be carrying the remaining cannibals to cross the river in step 5, depending on the number of cannibals we take in step 3.

I. The General Case Suggested by 林炳炎(2010)

林炳炎(2010) has read Fraley et al. (1966)'s solution and suggested that for the cases of boat capacity bigger than 3, and the number of missionaries and cannibals is bigger than the boat capacity by one, the best method is using 7 steps.

In my study, we could see that 林炳炎's(2010) suggestion is not fully correct. We have cases 3 to 5 ,suggesting that the best solution is using 5 steps. And in our proposition, we could see that the case would be true for the cases when B got larger. Therefore, we see that more examples and propositions need to be done before we propose any conclusions. This is also what I, as a prospective teacher, need to bear in mind that in teaching our students mathematics games. We also teach them to challenge and find ways to get to the truths by their own, not just only memorizing what the books have written. It is how we use the mathematical way to teach students as suggested by 馮振業(1997), that they think in a "mathematics" mind.

VI. Teaching Implication

A. Introduction

According to Hill, Ball and Schflling(2008), teaching is further classified into the subject knowledge and pedagogical. The subject knowledge of math is explicit in my mind, which is the universal way to present math problems and the math theories but pedagogically is how to let students have metacognitive thinking, as well as making them enjoy learning – can be observed with their on-task behavior. It is about the way that we lead students to think in a mathematical way, which is using reasons to support the different math theories or in math game. In solving math games like this missionaries and cannibals' problem, I believe students should be examined and fostered in their way of solving the problem, in which I would demonstrate by teaching Polya's problem-solving method.

In the following, I am going to first review some literatures on the usage of math game to increase on-task behavior, then design a lesson plan on the usage of Polya's method to solve the missionaries and cannibals' problem. After that, I would conduct it with a primary 6 student.

B. Literature Review – Advantages of Teaching Mathematical Games to Improve on-task Behavior

Bragg (2012) suggested that Math games could help to increase students' on-task behavior. They gave examples of visual representations of problems to help students understand the problem and give them ways to solve other daily life problems with the skill of making graphs to aid understanding. As games are diverse, it also provides them with opportunities to operate at different levels of thinking, in which I think we could use Bloom taxonomy to distinguish what they had learnt. The writers also pointed out that taking turns, watching opponents play ,and discussion of games happen in game teaching, for which I could see that their interpersonal skills may be trained. While Sullivan et al. (2009) said giving answers only in game playing is not good. I see this as a process, and students should show their thoughts out.

Some teachers have adopted game playing, and they gave their results. 陳綵菁 et al. (2013) said games help students understand math concepts like divisions. I would take her approach by letting students come out in front of the class to be cannibals and missionaries to cross the other side of the river, which could help them understand the game at the beginning. Getting them to understand the problem first is essential, as 黃家鳴(2003) explained that Hong Kong students are good at copying formulas to solve math, but not even understanding the problem. It is also noteworthy that 梁興強(2008) found out Hong Kong students are worse at problem-solving when comparing to Singaporean and Japanese kids.

C. Group of Students to Teach and the Rationale

The same topic could be taught to every type of students, just depending on how to adjust the way of letting them solve the problem.

The level of difficulty should be a little bit above students' ability, let's say as the zone of proximal development (Ciccarelli & White, 2015). It synchronizes with Hiebert and Weame's (1997) stating that when teachers pose higher-order tasks, students have been found to have more top achievements.

I designed this lesson plan for a group of primary 6 students ,and I find a primary 6 student to carry out this because according to Piaget (Ciccarelli & White, 2015), they are in the age of concrete operational period, in which they could gather new ideas to more sophisticated level. They would also be admitted to secondary schools, so I think getting them to understand some methods of problem-solving could well equip them to study in secondary schools, where higher order thinking is needed instead of mere memorization.

D. Rationale of letting Students to Understand River-crossing Games

For the sake of the missionaries and cannibals' problem is fun, as well as it is a daily life problem, I would like to teach my students about it. Some may said that there could be lots of things to teach them, but I think river-crossing problem is easy to understand, and the materials are easy to be prepared for a teacher, owing to what I heard from the honour project presentation of my classmates, that the teachers are very busy and do not have enough time to prepare the teaching materials.

In the sections above I have mentioned that math games could help students to acquire the skills needed for future development, I would like to explain in detail here. The Education Bureau (2000) stated that one of the aims of teaching math is to enhance the attitudes of students towards the learning of mathematics. I believe the co-construction of students to understand the problem together, that they come out to role-play the condition of river-crossing, and they discuss on how to find the "fewest steps" and appreciate the life application of math could well help build up this. We, as teachers, not only teach within the curriculum, if we have time, we could pursue our students' knowledge to some creative daily life problems to train their way of mathematical thinking.

Moreover, 梁貫成 et al. (1999) pointed out that though Hong Kong students got good results in TIMSS and they claimed that they love math, they did not get enough confidence when compared to students in other countries. Although I believe it may be because of the Chinese culture that we do not praise ourselves so much, students still need more confidence in order to achieve more. Therefore, in the lessons, students are asked to appreciate themselves in self-assessment and praise their peers in peer assessments. I would also congratulate them too.

E. Lesson Plan

Please refer to Appendix E for the lesson plan.

The focus of my lesson plan is that I do not teach general formula, where I would let students derive their method, as per the problem that Sulllivan et al. (2009) suggested, one of the issues of many teachers is that they only teach single ways to solve a problem.

The time for the lesson plan would be a double lesson as I would like to conduct the lesson after the exam period for the enhancement of math skills.

The Polya way of solving problems would be used ,and the steps are cited from 黃家鳴(2003), 梁興強(2008) and 香港教育城有限公司 (2017).

The way of guidance for the teacher would be in individual groups, not mass teaching. The teacher would observe around and provide some concrete tools like rubbers to represent the missionaries, for weaker students to use and think of how to solve the problems.

黃家鳴(2003) quoted the ways of Polya. I am writing the lesson plan as follows, with a double lesson of 60 minutes. For more details, you could go to the appendix to read it through.

First, we need to understand the problem, so I would use the first 10 minutes for direct instructions introducing the history and how to play. Then, I would give another 10 minutes for them to use their own words to describe the game as well as getting students to come out and role play, which could aid their understanding. It is also a student to student interaction that I would ask the audience students to interact with the role play students to see if they are performing the river-crossing conditions correctly. At the same time, the teacher would draw diagrams of the cases that they are acting and ask students if there are better ways to explain the situations, which could scaffold the students to find out the shortest path and represent it. Maybe dots and triangles could be used as an easier way for students to understand for teaching purpose, instead of using numbers beside “m” and “c” to show the number of people.

Then, they need to set up methods to solve. The teacher would give them worksheets to fill in during the 20 minutes group discussion. They do not need to draw diagrams like the teacher, that they could use their own words to do.

In the final section, the students are required to come out and present their ways of solving. It would be done in 20 minutes. Students also need to fill in self and peer evaluation form. 梁志強(2011)’s analytic rating scale for problem-solving is used. In the process, though I would like to find the methods with the fewest steps, I would praise the students with their attitudes and the mathematical thinking represented by graphs, lists or other things.

All in all, I would like students to think innovatively on how to solve the problems.

F. <u>Theories of Learning – Bloom Taxonomy</u>

The Bloom Taxonomy (Ciccarelli & White, 2015) is used. The understanding of how to play the game is knowledge-based. While finding their ways to represent the process of using the fewest steps to cross the river, it is at the creativity-based.

G. Aim of Research

The objective of the research is as follows.

1. Find out if teaching Polya steps of problem-solving by using the missionaries and cannibals problem is suitable for primary kids.
2. If it is suitable, what modifications could be used to make the teaching and learning better.

H. Research Participant

The participant is a primary 6 student, who has moderate result and is hard-working. She is also obedient in class. She is called Yuet in the following.

I. Research Methodology

The student was taught on the Polya method of problem-solving using the steps of the lesson plan. However, some minor modifications are done. First, there would be no role-play as there were not any classmates to do the role play with her. I used rubbers to represent missionaries and candies to represent cannibals. There were also no groupmates to discuss the issue with her. I answered her questions concerning what the possible methods are to use and questioned her back if her chosen method is useful. For example, in her drawing of the diagram, I asked her how she drew. She said she was going to use different colors to represent that.

After she had finished the steps, she filled in the self-assessment form. A mini interview is also done with the questions as follow.

1. Did you play math game in the lesson before?
2. How do you feel of this game?
3. Would you participate more if the lessons are taught in this way? How often would you like it to be?
4. How do you think of the level of difficulty of the game?
5. Do you feel capable of doing the problem-solving? Or any more aids should be provided by the teacher?

J. Result and Discussion

a. From the Student's view

One thing that is good for my research is that I could go into in-depth interview with the kid.

She said she had played the missionaries and cannibals game before and she found the game fun. She will also participate more if math lessons are conducted in this way. She viewed the level of difficulty as moderate. She felt capable of finding out the correct answer. She also explained it is not common for them to learn to think by themselves, so it would be good if more explanations could be provided from the teacher. She further said it was easy for her to carry out the process, but not the consideration of which method to use. She also appreciated the tools of models, which is one of the problem-solving approach, to help her try out the river-crossing condition.

b. **<u>My observation</u>**

Based on what Yuet explained, I believed more examples should be shown to students on utilizing various methods to do problem-solving questions, on listing, finding the sequence, backward thinking, use models, trial and error, drawing diagrams, simplify the question and direct reasoning as she found it challenging to decide which method to use.

It is discovered that she could make use of different colors and shapes to represent the missionaries and cannibals without my guidance, so she is capable to express ideas. Her difficulty is only on which method to use.

She also had the problem of representing the steps in an organized manner. She used an eraser to erase the missionaries and cannibals and draw them again to show the travelling instead of drawing a new diagram to represent each step. Therefore, I taught her to delineate more clearly in the review steps so she could be clear of how many steps she used and with the process shown. Moreover, she also forgot the backward action of travelling. If the steps are drawn more clearly, she could trace back which phase got wrong. Therefore, for future teaching, I believe during the discussion, I should go into the groups and review with them the clarity of steps. Yuet is tired of drawing all the steps too, so that more encouragement could be used on the teaching.

K. Research limitation

The limitation of the research is that I could only focus on one kid, which is not representing all the kids.

One of the other advantages is that the student is moderate, so it could reflect a bit of the level that primary students could achieve.

As I only tested on one student, so it could not reflect how group work could be done in the classroom. I hope soon when I become a teacher, I could try to see the implications of group work on problem-solving.

L. A Small Sum-up

In conclusion, let me quote what Sullivan et al. (2009) have said that it is the teacher who understands the problem first, so that they could teach well. During my process of finding ways to solve the missionaries and cannibals problem, and design lesson plans and think of how to solve, the whole process is helping me to cope with my teacher development.

VII. Conclusion

Overall, I have used Dijkstra's algorithm, Fraley et al. (2006) method to solve the river-crossing problem of missionaries and cannibals. I have also adapted and adopted the ways of finding trends from Fraley et al. (1966) method to design a lesson plan for the primary 6 students. In addition, a mini research is conducted to check if the teaching of river-crossing using Polya's method of problem-solving is suitable for primary 6 students, and the result is positive.

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Appendix A

The Missionaries and Cannibals Problem – How to Play

This game explores how we could get the missionaries and cannibals to cross to the other side of the river, with a boat which could carry 2 people at most at a time in addition to we need one person to paddle it back and forth. Our goal for the game is to get all the missionaries and cannibals to arrive the other side of the river, and there is a big restriction that the number of cannibals should not exceed the number of missionaries on either side of the river, or the cannibals will eat the missionaries. It has no problem on the boat though, as there are only 2 people allowed on the boat, in which the combination is restricted to 2 missionaries, 2 cannibals or 1 missionary accompanying 1 cannibal.

However, the difficulty of the game could be altered if we change the boat capacity and the number of missionaries and cannibals.

Appendix B

Dijkstra's Algorithm in Solving the Wolf, Cabbage and Goat problem

Shiu and Ling (2012) have introduced the Dijkstra's algorithm to help us deal with graph problems to start from one vertex to the other vertex to find the shortest path. This algorithm helps as it could keep track of pieces of information and results in different walks.

Alvis (2014) has made use of Dijkstra's algorithm to solve the Wolf, Cabbage and Goat problem. This is a problem in which we have the wolf, cabbage, goat and farmer who need to cross the river with a boat which has a capacity of 2 people. Only the farmer could paddle, and the wolf would eat the goat, the goat would eat the cabbage. We need to find the shortest path.

1. Let W, G, C and F denote wolf, garbage and farmer respectively.
2. Write down all the possible states as follows. The possible states would be the vertices of the graph. We decide the starting side of the river is left-hand side.

   V1: [WGCF |] (Start)

   V2: [| WGCF] (Goal)

   V3: [WCF | G]

   V4: [G | WCF]

V5: [WGF | C]

V6: [C | WGF]

V7: [GCF | W]

V8: [W | GCF]

V9: [GF | WC]

V10: [WC | GF]

3. The following list illustrates the edges that we would use, which would be used to construct a graph. The edges are the possible paths from one vertex to another. An undirected graph would be constructed. However, we aim to get all the missionaries and cannibals to the other side of the river, so we would not go back to the previous vertices.

   The edges:

   V1 → V10

   V2 → V9

   V3 → V6, V8, V10

   V4 → V5, V7, V9

   V5 → V4, V8

   V6 → V3, V7

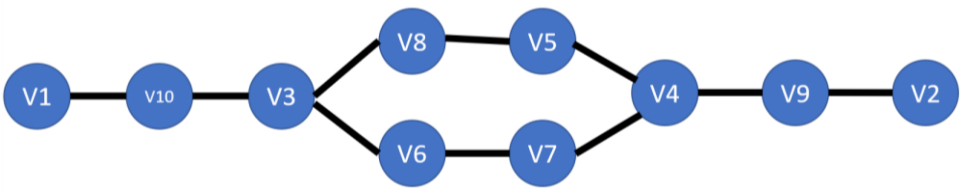
   V7 → V4, V6

V8 → V3, V5

V9 → V2, V4

V10 → V1, V3

4. Construct a graph to put in all the vertices and edges as follows.



5. Construct a form to include vertices into 3 groups, which are I(visited), II (current) and III (unvisited).

6. Put the starting vertex into II (current).

7. Put all the other vertices as III (unvisited).

8. Calculate the tentative distances from the current vertex to all the other vertices and write down the corresponding numbers in the tendency distance (abbreviated as T.D.) column. We define each edge as a step and count the distance as 1. Note that the ∞ sign would be used to represent the unknown path to different vertices.

	I(visited)	II(current)	III(unvisited)								
Vertex		V1	V2	V3	V4	V5	V6	V7	V8	V9	V10
T.D.		0	∞	∞	∞	∞	∞	∞	∞	∞	1

9. Move the II (current) vertex to I(visited).

10. Choose the vertex with the smallest tendency distance to move from III (unvisited) to II(current) to become the new current vertex. In the next step, we choose V10 as the next current vertex as it is the only reachable shortest path we could choose.

	I(visited)	II(current)	III(unvisited)							
Vertex	V1	V10	V2	V3	V4	V5	V6	V7	V8	V9
T.D.	0	1	∞	2	∞	∞	∞	∞	∞	∞

11. The steps are repeated until all vertices are visited. We overwrite the tentative distances whenever we find a shorter path and continue the process until all vertices are visited.

	I(visited)		II(current)	III(unvisited)						
Vertex	V1	V10	V3	V2	V4	V5	V6	V7	V8	V9
T.D.	0	1	2	∞	∞	∞	3	∞	3	∞

	I(visited)			II(current)	III(unvisited)					
Vertex	V1	V10	V3	V6	V2	V4	V5	V7	V8	V9
T.D.	0	1	2	3	∞	∞	∞	4	3	∞

	I(visited)				II(current)	III(unvisited)				
Vertex	V1	V10	V3	V6	V8	V2	V4	V5	V7	V9
T.D.	0	1	2	3	3	∞	∞	4	4	∞

	I(visited)					II(current)	III(unvisited)			
Vertex	V1	V10	V3	V6	V8	V5	V2	V4	V7	V9
T.D.	0	1	2	3	3	4	∞	5	4	∞

	I(visited)						II(current)	III(unvisited)		
Vertex	V1	V10	V3	V6	V8	V5	V7	V2	V4	V9
T.D.	0	1	2	3	3	4	4	∞	5	∞

	I(visited)							II(current)	III(unvisited)	
Vertex	V1	V10	V3	V6	V8	V5	V7	V4	V2	V9
T.D.	0	1	2	3	3	4	4	5	∞	6

	I(visited)								II(current)	III(unvisited)
Vertex	V1	V10	V3	V6	V8	V5	V7	V4	V9	V2
T.D.	0	1	2	3	3	4	4	5	6	7

	I(visited)									II(current)	III(unvisited)
Vertex	V1	V10	V3	V6	V8	V5	V7	V4	V9	V2	
T.D.	0	1	2	3	3	4	4	5	6	7	

	I(visited)										II(current)	III(unvisited)
Vertex	V1	V10	V3	V6	V8	V5	V7	V4	V9	V2		
T.D.	0	1	2	3	3	4	4	5	6	7		

Therefore, the two possible paths are:

- V1 → V10 → V3 → V8 → V5 → V4 → V9 → V2
- V1 → V10 → V3 → V6 → V7 → V4 → V9 → V2

Appendix C

The Process of Proving a More Interesting Problem of Missionaries and Cannibals Using Dijkstra's Algorithm (the weight of cannibals is 2 and that of missionaries is 1)

	I	II	III												
Vertex		A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.		0	2	4	3	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I	II	III											
Vertex	A	K1	E1	I1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	4	3	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I		II	III											
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I			II	III										
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I(visited)				II	III									
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	∞	∞	∞	∞	∞	∞	∞	∞	∞

	I					II	III								
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	11	∞	∞	∞	∞	∞	∞	∞	∞

	I						II	III							
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	11	13	∞	∞	∞	∞	∞	∞	∞

	I							II	III						
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	11	13	16	∞	∞	∞	∞	∞	∞

	I								II	III					
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	11	13	16	18	∞	∞	∞	∞	∞

	I										II	III			
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	11	13	16	18	20	24	∞	∞	∞

	I(visited)											II	III		
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	11	13	16	18	20	24	25	26	∞

	I(visited)												II	III	
Vertex	A	K1	I1	E1	K2	G	E2	J	I2	F1	H	D	J2	F2	B
T.D.	0	2	3	4	5	9	11	13	16	18	20	24	25	26	29

Appendix D

A. **Case 2 (B = 5, m = c = B + 1 = 6)**

<u>M1</u>:

(Start)		6m 6c	To 5c
S1			
S2	5c	6m 1c	Back 1c
S3	4c	6m 2c	To 2c
S4	6c	6m	Back 1c
S5	5c	6m 1c	To 5m
S6	5m 5c	1m 1c	Back 1c + 1m
S7	4m 4c	2m 2c	To 2c + 2m
(Goal)	6m 6c		

M2(A):

(Start)		6m 6c	To 5c
S1			
S2	5c	6m 1c	Back 1c
S3	4c	6m 2c	To 4m
S4	4m 4c	2m 2c	Back 1c + 1m
S5	3m 3c	3m 3c	To 2c + 3m
S6	6m 5c	1c	Back 1c
S7	6m 4c	2c	To 2c
(Goal)	6m 6c		

Again, step 5 could be varied as below.

M2(B):

S5	3m 3c	3m 3c	To 3m
S6	6m 3c	3c	Back 1c
S7	6m 2c	4c	To 4c
(Goal)	6m 6c		

M2(C)

S5	3m 3c	3m 3c	To 1c + 3m
S6	6m 4c	2c	Back 1c
S7	6m 3c	3c	To 3c
(Goal)	6m 6c		

M3(A)

(Start) S1		6m 6c	To 2c + 2m
S2	2m 2c	4m 4c	Back 1c + 1m
S3	1m 1c	5m 5c	To 5m
S4	6m 1c	5c	Back 1c
S5	6m	6c	To 5c
S6	6m 5c	1c	Back 1c
S7	6m 4c	2c	To 2c
(Goal)	6m 6c		

The steps from step 5 onwards could vary.

M3(B)

S5	6m	6c	To 4c
S6	6m 4c	2c	Back 1c
S7	6m 3c	3c	To 3c
(Goal)	6m 6c		

M3(C)

S5	6m	6c	To 3c
S6	6m 3c	3c	Back 1c
S7	6m 2c	4c	To 4c
(Goal)	6m 6c		

M3(D)

S5	6m	6c	To 2c
S6	6m 2c	4c	Back 1c
S7	6m 1c	5c	To 5c
(Goal)	6m 6c		

B. **Case 3 (B = 7, m = c = B + 1 = 8)**

<u>M1</u>:

(Start)		8m 8c	To 7c
S1			
S2	7c	8m 1c	Back 1c
S3	6c	8m 2c	To 2c
S4	8c	8m	Back 1c
S5	7c	8m 1c	To 7m
S6	7m 7c	1m 1c	Back 1c + 1m
S7	6m 6c	2m 2c	To 2c + 2m
(Goal)	8m 8c		

<u>M2</u>:

(Start)		8m 8c	To 7c
S1			
S2	7c	8m 1c	Back 1c
S3	6c	8m 2c	To 6m
S4	6m 6c	2m 2c	Back 1c +1m
S5	5m 5c	3m 3c	To 3c +3m
(Goal)	8m 8c		

M3(A)

(Start)		8m 8c	To 3c + 3m
S1			
S2	3m 3c	5m 5c	Back 1c +1m
S3	2m 2c	6m 6c	To 6m
S4	8m 2c	6c	Back 1c
S5	8m 1c	7c	To 7c
(Goal)	8m 8c		

M3(B)

S3	2m 2c	6m 6c	To 1c + 6m
S4	8m 3c	5c	Back 1c
S5	8m 2c	6c	To 6c
(Goal)	8m 8c		

So, a small conclusion could be drawn that method one is not a fitting method as it contains 7 steps for this case. In particular, step 3 in method 1 is not "smart" enough, in which we only carry 2 cannibals, and the boat capacity of 7 people is not fully utilized. Whereas, method 2 and 3 are equally good that they consist of 5 steps. Starting from case 3, M1 is no longer and excellent method, but M2 and M3 both yield the fewest 5 steps.

C. **Case 4 (B = 9, m = c = B + 1 = 10)**

M1:

(Start)		10m 10c	To 9c
S1			
S2	9c	10m 1c	Back 1c
S3	8c	10m 2c	To 2c
S4	10c	10m	Back 1c
S5	9c	10m 1c	To 9m
S6	9m 9c	1m 1c	Back 1c + 1m
S7	8m 8c	2m 2c	To 2c + 2m
(Goal)	10m 10c		

M2:

(Start)		10m 10c	To 9c
S1			
S2	9c	10m 1c	Back 1c
S3	8c	10m 2c	To 8m
S4	8m 8c	2m 2c	Back 1c + 1m
S5	7m 7c	3m 3c	To 3c + 3m
(Goal)	10m 10c		

M3(A):

(Start)		10m 10c	To 4c + 4m
S1			
S2	4m 4c	6m 6c	Back 1c + 1m
S3	3m 3c	7m 7c	To 7m
S4	10m 3c	7c	Back 1c
S5	10m 2c	8c	To 8c
(Goal)	10m 10c		

Again, there is room for derivation in the last few steps. Please refer to the following.

M3(B):

S3	3m 3c	7m 7c	To 1c + 7m
S4	10m 4c	6c	Back 1c
S5	10m 3c	7c	To 7c
(Goal)	10m 10c		

M3(C):

S3	3m 3c	7m 7c	To 2c + 7m
S4	10m 5c	5c	Back 1c
S5	10m 4c	6c	To 6c
(Goal)	10m 10c		

For case 4, again M2 and M3 yield the best 5 steps, but not M1.

D. **Case 5 (B = 11, m = c = B + 1 = 12)**

<u>M1</u>:

(Start)		12m 12c	To 11c
S1			
S2	11c	12m 1c	Back 1c
S3	10c	12m 2c	To 2c
S4	12c	12m	Back 1c
S5	11c	12m 1c	To 11m
S6	11m 11c	1m 1c	Back 1c + 1m
S7	10m 10c	2m 2c	To 2c + 2m
(Goal)	12m 12c		

<u>M2</u>:

(Start)		12m 12c	To 11c
S1			
S2	11c	12m 1c	Back 1c
S3	10c	12m 2c	To 10m
S4	10m 10c	2m 2c	Back 1c +1m
S5	9m 9c	3m 3c	To 3c +3m
(Goal)			

M3(A):

(Start)		12m 12c	To 5c + 5m
S1			
S2	5m 5c	7m 7c	Back 1c + 1m
S3	4m 4c	8m 8c	To 3c +8m
S4	12m 7c	5c	Back 1c
S5	12m 6c	6c	To 6c
(Goal)	12m 12c		

From step 3 on, there could be a varied combination of crossing methods.

M3(B)

S3	4m 4c	8m 8c	To 2c +8m
S4	12m 6c	6c	Back 1c
S5	12m 5c	7c	To 7c
(Goal)	12m 12c		

M3(C)

S3	4m 4c	8m 8c	To 1c +8m
S4	12m 5c	7c	Back 1c
S5	12m 4c	8c	To 8c
(Goal)	12m 12c		

M3(D)

S3	4m 4c	8m 8c	To 8m
S4	12m 4c	8c	Back 1c
S5	12m 3c	9c	To 9c
(Goal)	12m 12c		

For case 5, M2 and M3 yield 5 steps too.

Appendix E

Lesson plan and materials

Year level / Class: Primary 6

Topic: Polya Problem-solving Method – Missionaries and Cannibals Problem

Length of time: 60 min

Learning objectives:

Students should be able to

- ✓ Describe the missionaries and cannibals game using role play and word explanation.
- ✓ Investigate the method with fewest steps and present the ideas to the class using Polya method of problem-solving of the missionaries and cannibals problem (with boat capacity as even number and the number of missionaries and cannibals be the boat capacity plus one).

- Learning outcomes for students with diverse needs – after this lesson:
  The students are divided into heterogeneous groups of 4. The stronger students could contribute more into the process of finding a suitable method for solving this question, while the weaker students could try to follow the chosen method to find out the correct answer.

Previous knowledge:

- ✓ Solve math questions using Polya method of problem-solving.
- ✓ Record with algebraic symbols.

Blackboard / Whiteboard Planning:

Polya method 1. Understand the problem 2. Derive ways to solve the problem 3. Solve the problem 4. Review the method	The drawing of the teacher of the situations when the students are coming to act out the situations

Teaching & Learning resources / aids:

- Worksheet
- Cubes/ other materials which are accessible for illustration of river-crossing
- Word cards for generating their questions
- Video from HKEdCity

Lesson procedures:

	Content	Resources	Time	Interactions	Rationales
1	*<u>Introduction: Understanding the Missionaries and Cannibals Problem</u>*	/	10’	T → ss	1. Students get the factual background

	The teacher introduces the history of the missionaries and cannibals problem and explains the game rules.				of the problem.
2	*Understanding and Propose Ways to Solve* • Students come out and role play the situations. E.g. 3 missionaries and 3 cannibals(With boat capacity of 2), 5 missionaries and 5 cannibals (with boat capacity of 4)	/	10’	ss → ss	1. Kinetic learning to aid the understanding of the river-crossing process. 2. The students would point out whether their classmates are acting the right conditions. The teacher won’t give her comments until there is no other students point out that to induce peer interaction.
3	*Group Discussion* • 4 students in a group • The teacher introduces the Polya method of problem-solving (1. Understanding the problem. 2. Devise a plan. 3. Carry out the plan. 4. Review the method). The teacher also explains the possible ways of problem-solving (Listing out, finding the sequence, backward thinking, trial and error, drawing diagrams, simplify the question, direct reasoning or any method the students could think of) - E.g. Use models to represent the missionaries and cannibals (the teacher could ask the students how to use the models first to	✓ WS ✓ Video	30’	ss ←→ ss	1.Students engage in active discussion of river-crossing. 2.Evaluate self and others on the process of problem-solving. The main focus is not on the use of Polya method. Polya could be an excellent method for use, but students

	induce their thinking. Unless they could not think of methods, the teacher tries to scaffold them by giving some examples or tips.) • Every student should contribute to the ways and representation of how to find the shortest path 1. One student mark on the worksheet 2. One student checks the process of marking 3. One student reminds if there is any missing of the discussion details 4. One student as the material holder, for getting the cubes from the teacher as the material to aid understanding of process of river-crossing • Group presentation and students summarize the methods of river-crossing. • Self and peer assessment				could derive their way.
4	<u>*Consolidation- General Rules for Missionaries and Cannibals River-crossing*</u> • Pair Discussion of general methods (Expected answers: 1. One cannibal or one missionary and cannibal could be used to cross the boat back to the starting side. 2. Method 1: Step 1: send B cannibals to the destination side of the river. Step 2: Send 1 cannibal back. Step 3: Send B – 1 missionaries to the destination side of the river. Step 4: Send 1 cannibal	✓ Word cards	10’	ss ←→ ss ss←→T	1. The students could summarize and create general methods for problem-solving

	and 1 missionary back. Method 2: Step 1: send $\frac{B}{2}$ cannibals and missionaries to the destination side of the river. Step 2: Send 1 cannibal and 1 missionary back. • Share to the class • Create their own river crossing problems				
5	*Conclusion* • Students are randomly chosen to repeat the Polya steps of problem-solving	/			1. Revising the Polya steps of problem-solving
6	*Homework assignment* • Solving river crossing questions involving missionaries and cannibals • Creating their questions and solve them	✓ WS		----	1. Practice their ways of problem-solving and create new questions for themselves

Formative Assessment strategies:

- Direct observation: During the sharing time with classmates, the teacher walks around. The teacher gives feedback for the students
- Formal strategies:
  1. The WS for applying the way of problem-solving would be collected.

There are _____missionaries and _____cannibals who are aiming to cross to the other side of the river with the boat capacity of _____. Find the fewest steps for them to cross.

Peer-assessment form (Please tick ✓) Name of groupmate:______________________	0 mark	1 mark	2 marks
<u>Understanding the problem</u> I think during the discussion, he/she 0 mark: Does not understand the problem 1 mark: Understands that we need to fewest steps to cross the river (Partly understand) 2 marks: Understands all the requirement of the problem-solving question (the boat capacity burden and find out the fewest steps involved)			
<u>Choosing the suitable problem-solving method</u> 0 mark: Do not have any idea of which method to use 1 mark: Try to think of some methods and suggest to the group 2 marks: Could think of the pros and cons of different methods and suggest to groupmates			
<u>Finding the answer</u> 0 mark: could not solve the question because of choosing the wrong method 1 mark: make use of the method and find out an answer, though it does not contain the fewest steps 2 marks: Could make use of the chosen method to find out the answer with the fewest steps			
He or she is respectful and participate actively in the discussion. (Please rate from 1 to 5. 1 is totally disagree, 2 is a bit disagree, 3 is neutral, 4 is a bit agree, 5 is totally agree)	Mark:___________		

XXX Primary school

Mathematics

The Missionaries and Cannibals River-Crossing Game Homework

Group number: _____ Class: ______ Date: _______________________

A. Check Your Understanding

Put 1 – 4 in the following brackets to represent the steps of Polya problem-solving method.

( ) Set up a method

( ) Review

( ) Understanding the problem

( ) Carry out the method

B. Polya's method

Answer the following questions

Question: There are 8 missionaries and 8 cannibals who are aiming to cross to the other side of the river with the boat capacity of 7. Find the fewest steps for them to cross.

1. Understanding the problem (Write down your understanding of the goal of this problem solving.)
2. Set up methods to solve. Which way of problem-solving methods would you use? (Listing out, finding the sequence, backward thinking, use models, trial and error, drawing diagrams, simplify the question, direct reasoning, etc)
3. Carry out the procedures clearly.

4. Revise your plan. How is your chosen problem-solving method?What general methods of river-crossing method you have used compared to the lesson?

C. Little Designer

Design your own river-crossing question and solve it. (Condition: the boat capacity should be an even number, while the number of missionaries and cannibals be the boat capacity plus one)

Question: There are 8 missionaries and 8 cannibals who are aiming to cross to the other side of the river with the boat capacity of 7. Find the fewest steps for them to cross.

1. Understanding the problem (Write down your understanding of the goal of this problem solving.)
2. Set up methods to solve. Which way of problem-solving methods would you use? (Listing out, finding the sequence, backward thinking, use models, trial and error, drawing diagrams, simplify the question, direct reasoning, etc)
3. Carry out the procedures clearly.
4. Revise your plan. How is your chosen problem-solving method?What

general methods of river-crossing method you have used compared to the lesson?

A. Check your Understanding

Put 1 – 4 in the following brackets to represent the steps of Polya problem-solving method.

(2) Set up a method

(4) Review

(1) Understanding the problem

(3) Carry out the method

B. Polya's method

Answer the following questions

Question: There are 8 missionaries and 8 cannibals who are aiming to cross to the other side of the river with the boat capacity of 7. Find the fewest steps for them to cross.

1. Understanding the problem (Write down your understanding of the goal of this problem solving.)
2. Set up methods to solve. Which way of problem-solving methods would you use? (Listing out, finding the sequence, backward thinking, use models, trial and error, drawing diagrams, simplify the question, direct reasoning, etc)
3. Carry out the procedures clearly. (The following is just one of the possible ways to solve this question. Students could have their own representation.)
4. Revise your plan. How is your chosen problem-solving method?What

general methods of river-crossing method you have used compared to the lesson?

C. Little Designer(Free Response)

Design your own river-crossing question and solve it. (Condition: the boat capacity is an even number, while the number of missionaries and cannibals be the boat capacity plus one)

Question: There are 8 missionaries and 8 cannibals who are aiming to cross to the other side of the river with the boat capacity of 7. Find the fewest steps for them to cross.

1. Understanding the problem (Write down your understanding of the goal of this problem solving.)
2. Set up methods to solve. Which way of problem-solving methods would you use? (Listing out, finding the sequence, backward thinking, use models, trial and error, drawing diagrams, simplify the question, direct reasoning, etc)
3. Carry out the procedures clearly.
4. Revise your plan. How is your chosen problem-solving method? What general methods of river-crossing method you have used compared to the lesson?

Appendix F

Consent form

THE EDUCATION UNIVERSITY OF HONG KONG
Department of Mathematics and Information Technology

CONSENT TO PARTICIPATE IN RESEARCH

Exploring ways of proving the missionaries and cannibals problem and the Teaching implication

I ____________________ hereby consent to participate in the captioned research supervised by Suen Chun Kit Anthony and conducted by Ng Kam Ling, who are staff / students of Department of Mathematics and Information Technology in The Education University of Hong Kong.

I understand that information obtained from this research may be used in future research and may be published. However, my right to privacy will be retained, i.e., my personal details will not be revealed.

The procedure as set out in the **attached** information sheet has been fully explained. I understand the benefits and risks involved. My participation in the project is voluntary.

I acknowledge that I have the right to question any part of the procedure and can withdraw at any time without negative consequences.

Name of participant ____________________

Signature of participant ____________________

Date ____________________

1

INFORMATION SHEET

Department of Mathematics and Information Technology

You are invited to participate in a project supervised by Suen Chun Kit Anthony and conducted by Ng Kam Ling, who are staff / students of the Department of Mathematics and Information Technology in The Education University of Hong Kong.

The aim of the study is to find out the effectiveness of using Polya's method of problem solving in students using the missionaries and cannibals problems. You are chosen for this research as you are a primary 6 student, which fits the lesson plan I have designed.
I would include one participant from this study, and I got your detail as I am your private tutor. You are expected to propose ways to solve the missionaries and cannibals' river crossing problem. You are required to describe your ways to solve the problem and write it out. Then, you would review her problem-solving method with me. The research is done at your home and after school for one time. The review would be done in interview and questionnaire to ask her if you had played similar games before and whether you think the game has any positive or negative impacts on her learning. The total expected time would be 60 minutes. The benefits for you is to learn a problem-solving method.
There is no potential risk for joining this research.
Your participation in the project is voluntary. You have every right to withdraw from the study at any time without negative consequences. All information related to you will remain confidential, and will be identifiable by codes known only to the researcher.
Your name would not be shown on the article. The worksheets you filled would be shown in my honour project and shown to my supervisor. It may also appear in the research depository of the Education University of Hong Kong.
If you would like to obtain more information about this study, please contact Ng Kam Ling at telephone number or their supervisor Suen Chun Kit Anthony at telephone number

If you have any concerns about the conduct of this research study, please do not hesitate to contact the Human Research Ethics Committee by email at hrec@eduhk.hk or by mail to Research and Development Office, The Education University of Hong Kong.
Thank you for your interest in participating in this study.

Ng Kam Ling
Principal Investigator

香港教育大學

數學與資訊科技學系

參與研究同意書

探討土人和僧人過河遊戲的解難方法和教學的啟示

本人 ________同意參加由孫俊傑負責監督,吳金玲執行的研究項目。她/他們是香港教育大學數學與資訊科技學系的學生/教員。

本人理解此研究所獲得的資料可用於未來的研究和學術發表。然而本人有權保護自己的隱私，本人的個人資料將不能洩漏。

研究者已將所附資料的有關步驟向本人作了充分的解釋。本人理解可能會出現的風險。本人是自願參與這項研究。

本人理解我有權在研究過程中提出問題,並在任何時候決定退出研究，更不會因此而對研究工作產生的影響負有任何責任。

參加者姓名: ______________________

參加者簽名: ______________________

日期: 8/4/2019

2

有關資料

探討土人和僧人過河遊戲的解難方法和教學的啟示

誠邀閣下參加孫俊傑負責監督，吳金玲負責執行的研究計劃。她/他們是香港教育大學數學與資訊科技學系的學生/教員。

本研究計劃的目的為透過與學生一起用波利亞的解難四步曲解決土人和僧人過河的問題，評估波利亞的解難四步曲的成效和教學啟示。本次教學設計的對象是小學六年級的學生，所以參與者也選定了小學六年級的學生。本次的研究參與者為一個小學六年級的學生，本人在私人補課中得悉這個參與者的資料。你將會就著土人和僧人過河遊戲，提出解難方法，本人會和你一起檢討不同的解難策略。本次的研究需時一小時，會在放學後進行。在解難過程後，你將會填問卷和接受訪問，看看成效和你的看法感受。是次研究並不為閣下提供個人利益，但是可以讓你學習不同的解難策略。

本次的研究沒有任何風險。

閣下的參與純屬自願性質。閣下享有充分的權利在任何時候決定退出這項研究，更不會因此引致任何不良後果。凡有關閣下的資料將會保密，一切資料的編碼只有研究人員得悉。

你的個人資料將會保密，只作研究員識別之用，但你完成的工作紙和訪問會發佈於畢業論文內，也有可能在香港教育大學圖書館展示。

如閣下想獲得更多有關這項研究的資料，請與吳金玲聯絡，電話 或聯絡她/他們的導師孫俊傑，電話

如閣下對這項研究的操守有任何意見，可隨時與香港教育大學人類實驗對象操守委員會聯絡(電郵: hrec@eduhk.hk； 地址:香港教育大學研究與發展事務處)。

謝謝閣下有興趣參與這項研究。

吳金玲
首席研究員

4

Appendix G

The worksheet that the student did

XXX Primary school

Mathematics

The Missionaries and Cannibals River-Crossing Game

Group number: _____ Class: ______ Date: ______________________

A. Write down your groups' steps of Polya method of problem solving as follows.

Question: There are 5 missionaries and 5 cannibals who are aiming to cross to the other side of the river with the boat capacity of 4. Find the fewest steps for them to cross.

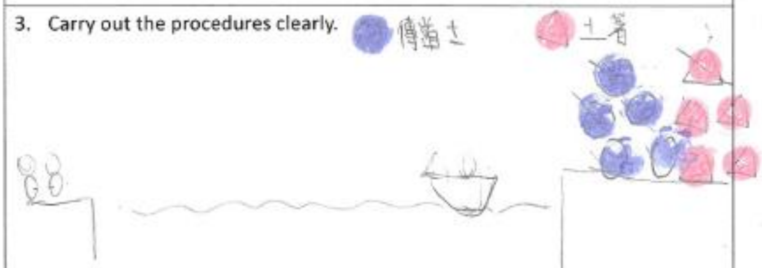
1. Understanding the problem (Write down your understanding of the goal of this problem solving.)

有五個傳道士和五個土著，要把他們過河而不會被土著吃掉

2. Set up methods to solve. Which way of problem-solving methods would you use? (Listing out, finding the sequence, backward thinking, use models, trial and error, drawing diagrams, simplify the question, direct reasoning, etc)

繪圖

3. Carry out the procedures clearly.



4. Revise your plan with your teacher. What do you think you have done well in the process? What could be improved?

B. Do you found any general methods of river-crossing.

	Self-assessment form (Please tick ✓)	0 mark	1 mark	2 marks
1.	**Understanding the problem** I think I...... 0 mark: Do not understand the problem 1 mark: Understand that we need to fewest steps to cross the river (Partly understand) 2 marks: Understand all the requirement of the problem-solving question (the boat capacity burden and find out the fewest steps involved)			✓
2.	**Choosing the suitable problem-solving method** 0 mark: Do not have any idea of which method to use 1 mark: Try to think of some methods and give a try 2 marks: Could think of the pros and cons of different methods and choose the method to try		✓	
3.	**Finding the answer** 0 mark: could not solve the question because of choosing the wrong method 1 mark: make use of the method and find out an answer, though it does not contain the fewest steps 2 marks: Could make use of the chosen method to find out the answer with the fewest steps			✓
4.	I enjoy doing problem-solving questions than normal lessons. (Please rate from 1 to 5. 1 is totally disagree, 2 is a bit disagree, 3 is neutral, 4 is a bit agree, 5 is totally agree)	Mark: 4		

Appendix H

The draft we drew together in the review session for her better representation of the process of problem-solving next time