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A Project entitled

Using Variation Theory to enhance the learning of "Fitting Shapes" of primary four students in Hong Kong

Submitted by

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Declaration

I, *LIU Yujia*, declare that this research report represents my own work under the supervision of *Dr. ZHANG Yuefeng Ellen*, and that it has not been submitted previously for examination to any tertiary institution.

Signature: ____________________

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10.05.2020

Abstract

Variation Theory (VT) was first raised by Professor Ference Marton in 1998, stating that learning implies seeing, perceiving, or experiencing critical aspects of the *object of learning (OL)* (Marton & Booth, 1997). When VT is applied in teaching, teachers can design examples and activities with reference to the patterns of variation to allow students to experience the variation and discern *critical features (CFs)* of the *OL*. Previous studies (Driver et al., 2015; Chiu & Bo, 2012; Ting et al., 2017; Leung et al., 2011) demonstrated how to apply VT to support the teaching of one single lesson in various disciplines. However, these studies did not mention how to arrange the teaching of CFs in multiple lessons on the same topic and how teachers guided the students in detail during the whole instruction.

This study aims at exploring the application of VT in enhancing students' learning of "fitting shapes" of primary four students in Hong Kong. This study adopted an action research approach. A total of 31 students, including 14 boys and 17 girls, participated in this study. With reference to the pre-test results, the author identified two CFs and designed organic lessons on the topic of "fitting shapes". In addition to showing a series of examples in classroom teaching and the corresponding patterns of variation, the study also demonstrated the interactive teaching process. Apart from collecting the observation data to analyze what VT practices enhanced the students' learning, the author collected post-test results and compared them with the pre-test results to see which parts of VT practices need to be further improved.

The findings of this study showed that the overall students' learning of "fitting shapes" had been enhanced. The teaching process with the application of four patterns of variation (contrast, separation, generalization, and fusion), the series of examples matched with CFs, and the appropriate teaching interaction facilitated students' learning. Nevertheless, to further enhance students' learning, the teacher may need to add more graphics types (shape, placement) in the examples and counterexamples so as to help students solve more complex situations of "fitting shapes".

Keywords: Variation Theory, mathematics in primary schools, fitting shapes, action research

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Last but not least, I would like to extend my appreciation to my beloved parents, Zheng and Lian, who are always there through my ups and downs, listening to my concerns and supporting me in pursuing my dream endlessly. Stepping into society, I wish I could also, in return, become their reliance in the future.

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1. Introduction

1.1 Research Background and Significance

The latest curriculum reform in Hong Kong primary school has been launched since the publication of the *Basic Education Curriculum Guide – Building on Strengths (Primary 1 – Secondary 3)* in 2002, focusing on providing comprehensive and balanced learning experiences for students (Curriculum Development Council, 2014). It introduced a flexible curriculum framework to facilitate the shift in school education, gradually steering from a teacher-centered teaching approach to a student-oriented learning approach (Curriculum Development Council, 2014). Teachers are suggested to adopt different strategies that can cater to individual needs, learning styles, interests, and abilities (Lam, 2002). Schools should develop the school-based curriculum most suited to the abilities and needs of students and continue to raise the quality of teaching and learning (Cheong Cheng, 2009). Under such a context of reform, many new theories of teaching and learning have been tried out by in-service teachers in Hong Kong. Variation Theory (VT) was one of them.

VT originally stemmed from Marton's (1986) empirical study of learning through the phenomenography research approach, aimed at identifying the possibilities that enable people to cultivate a powerful way of seeing or undergoing a particular phenomenon (Lam & Tsui, 2013). Offering a theoretical framework for exploring possible changes in experience and promoting learning and understanding (Bussey, Orgill, & Crippen, 2013), VT can be used to address individual differences in the classroom by allowing students to learn from their personal experiences and apply it to learning (as cited in Cheng, 2016). Over the years, researchers have applied the VT in several subject areas with examples (Hella & Wright, 2009; Runesson, 2005; Lam & Tsui, 2013; Tse, Marton, Ki, & Loh, 2007) around the world. Among the researches, VT was found to be a positive guiding role in the instructional designs and practical teaching of each subject, including mathematics.

Mathematics teaching in Hong Kong stresses the mastery of fundamental and connected concepts within the major field of knowledge (Education Bureau, 2017). Teachers usually use multiple examples to facilitate the students' acquisition of the concepts. This coincides with the principle of VT, which emphasizes the discernment of the new concepts and patterns through the experience of similarities and differences between various examples (Kullberg, Runesson, & Marton, 2017). Leung et al. (2011) showed several studies of successful application of VT in the teaching of different mathematical topics (including fractions, algebra, equations, etc.) in Hong Kong. All of these teaching cases (Leung et al., 2011) found out that students' learning effectiveness had been enhanced after experiencing the questions and activities which were designed with variation patterns.

However, in most of the past studies (Driver, Elliot, & Wilson, 2015; Chiu & Bo, 2012; Ting, Tarmizi, Bakar, & Aralas, 2017), researchers explored how to apply the VT and demonstrated its validity based on one single lesson of a particular subject. This study attempted to design and implement the teaching of a whole topic, including four lessons based on VT, demonstrating the application of VT in mathematics teaching from a more complete and continuous perspective. Taking advantage of the author’s practicum, the author hoped to conduct action research using the teaching of “fitting shapes” （圖形拼砌） as an example, so as to explore the effects and the limitations of VT in mathematics teaching.

The thesis consists of six parts. The author first reviews the literature both on VT and its relationship with mathematics education, then introduces the design of this action research and the process of applying VT in teaching. Finally, it reports results from analyzing the data and concludes with suggestions for further improvement.

2. Literature Review

In this section, the key concepts of VT and its main application process in teaching and learning are introduced. It also reviews the use of teaching strategies in VT and discusses the relationship between theory and mathematics education.

2.1 Variation Theory

Variation Theory was first raised by Professor Ference Marton in 1998, stating that learning implies seeing, perceiving, or experiencing critical aspects of the *object of learning* (OL) (Marton & Booth, 1997). The object of learning refers to “a specific insight, skill or capability that the students are expected to develop” (Marton & Pang, 2007, p.2). It consists of two aspects: the direct and indirect object of learning. So, teachers' focus should be on both: the content which the learners are trying to learn, as well as the expected capability or attitude which can be developed through learning (Marton & Tsui, 2004, p.19). To get command of the object of learning, students must experience the variation. Marton and Booth (1997) posited that qualitatively changed ways of experiencing something are the most advanced form of learning. The experience of differences is essential for learning to perceive the new thing. "Without variation, there is no discernment" (p.8), only when a specific aspect changes while all other aspects of the phenomenon remain invariant, can the learners discern the varying aspect (Bowden & Marton, 1998). In other words, experiencing the differences is fundamental to learning. This point of view was also supported by the constructivism. Luhmann Constructivist Theory of knowledge (1990) pointed out that, all of the knowledge formations are operated by the observation of a difference, thus composing a distinction, which is similar to the "discernment" mentioned by Marton and Booth (1997).

The application of VT in classroom teaching can be divided into several steps in teaching and learning. First of all, teachers ought to decide the *object of learning*, which is different from the learning objectives. Learning objectives are something that specified in the curriculum guide, aiming for the result of learning, while the *object of learning* points to the starting point of the learning journey (Lo, 2012). When identifying the *object of learning*, teachers are also expected to understand the internal structure and significance of it, such as the relationship between the whole and the parts. Secondly, teachers should direct students to further discover the *critical features* (CF) of the *object of learning*. The term *critical features* (CF) is used to describe as the "features and conditions necessary for learning" (Rundgren & Tibell, 2010, p.229). Lo (2012) believed that people's understanding of a particular object is established on our simultaneously focusing on certain features, which could be found out from literature review and sharing among teachers, or students' perspectives. Through identifying the critical features, teachers could understand the object of learning better and plan his or her teaching more appropriately and concretely. Thirdly, utilizing the variation of the critical features through contrast, separation, generalization, and fusion to assist students in learning the most valued contents. Teachers can use four patterns of variation mentioned above in their teaching to help their students to discern the critical features (Lo, 2012). In this part, students' participation is extremely important since the core of Variation Theory-based teaching is to let students experience the process of discerning the critical features via variation patterns.

The effectiveness of variation also relies on other teaching strategies. Among various subjects and topics, only when appropriate teaching strategies are employed can the students experience the pattern as intended. Suggested by Lo (2012), teachers can obtain feedback from their students timely in class and utilize students' differing views to let them experience more variation in understanding of the same object. Teachers could also make good use of students' counterexamples as the new teaching resources in variation. Moreover, teachers should prepare target teaching activities that facilitate the learning of critical features (Lo, 2012). For instance, asking proper guidance questions or organizing the classroom activities could provide ample opportunities (Lo, 2012) for students to experience the relevant patterns of variation so as to discern the critical features.

2.2 Variation Theory and Mathematics Education

Runesson (2005) has demonstrated the potentials of VT in providing insights into mathematics teaching and classroom learning through the analysis of concrete examples and data. Variation interaction is a strategic use of variation to interact with a

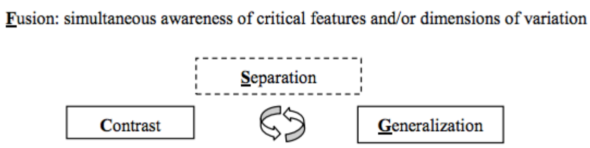


Figure 2.2.1: Pedagogical model (Leung, 2012)

mathematics learning environment for students to discern the mathematical structure (Leung, 2012). Refer to Leung (2012), four types of variation interaction, (contrast, separation, generalization, and fusion) which developed from the Marton's theory of variation, act together in a concerted way to bring about discernment. Leung (2012) proposed a pedagogical model (see figure 1) to show that the lessons should consist of a sequence of variation interactions in order to let students be aware of the critical features. She described it as "a convolution of contrast and generalization driven by separation fused together by simultaneous awareness of critical features" (Leung, 2012, p.436). These four kinds of variation interactions are mutually reinforcing.

Additionally, some scholars (Rittle-Johnson, Star, & Durkin, 2009; Schwartz & Bransford, 1998) have pointed out that using multiple examples in mathematics instruction could better support students' learning. Watson and Mason (2006) also mentioned that examples with systematic variation might help students to discern similarities and differences associated with the concepts. Kullberg et al. (2017) used the topic of '*solving equations with one unknown*' as an example to discuss the enacted sets of examples and tasks for assisting students in discerning the different aspects. In the lesson, several examples of equations with a different number of each side of the equal sign were discussed to enable students to discern that all parts of the equations must have the same value for the equal sign to work. Another task introduced two ways of solving one equation 3*x*+5=20 (iconic and numerical), in which the equation remained invariant while the presentation of the equation varied. Such a task made it possible for students to discern that *x* can also be represented as other objects. Besides, the examples of non-solvable equations for positive integers (3*x*+20=5 and 2*x*+3=3*x*+4) and the equations derived from the text problem have been provided in the latter part of the lesson.

1. $3\times4=12$
2. $12=3\times2\times2$
3. $12\times2=3\times2\times2\times2$
4. $3\times6=18\times2=36$ $3\times6\neq18\times2=36$

Figure 2.2.2: Examples of equations (Kullberg et al., 2017)

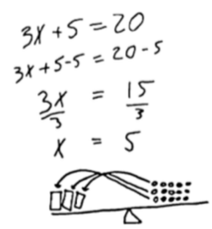


Figure 2.2.3:
Two ways of solving an equation (Kullberg et al., 2017)

Throughout the lesson, the tasks and the series of examples with variation could make it possible for the students to experience the differences as well as similarities in relation to critical aspects (Kullberg et al., 2017), which was helpful for students' learning.

Although the above literature explained how to add proper examples in the four types of variation interaction in mathematics teaching, they did not mention how to arrange the teaching of CFs in multiple lessons and how teachers guided the students in detail during the whole instruction. In addition to ascertaining what aspects should be discerned in the lessons and preparing a series of examples referring to VT, the teachers should also pay attention to how to carry out the teaching process of organic lessons, which are more than just the design of variation patterns. Therefore, the author conducted an action research during her practicum, hoping to find the usefulness and feasibility of VT through her individual teaching and supplement the above deficiencies of the past empirical studies.

3. Research Design

3.1 Research Nature

This study adopted action research approach, which is an organized investigation approach that enables people to find effective solutions to problems they meet in daily life, uses a continuing cycle of investigation to reveal the effectiveness and efficiency of people's work (Stringer, 2014). In education, action research is a model of research that has the improvement of practice (Parsons & Brown, 2002). According to Mills (2007), action research may help teachers to embrace action, progress, and reform. Through action research, teachers can keep reflecting on the effects of students' learning and their teaching systematically. In 2013, McAteer introduced that action research in the field of education could be carried out in more detail through the following five steps:

1. Starting off, and clarifying a research question.
2. What is the situation at present? And how can I find out?
3. What changes can I make?
4. Evaluate the effect of these changes.
5. Revisiting the original research question in light of your findings.

Taking this step model as a reference, the author recorded and discussed the action research in this thesis.

3.2 Research questions and methods

This research aimed to explore the effectiveness of using VT to enhance primary four students' learning in "fitting shapes" and find out what kinds of particular activities or teaching arrangements are helpful in practice. The following three questions were raised:

1) Was students' learning of "fitting shapes" enhanced by the use of VT?

2) If yes, what VT practices enhanced students' learning of "fitting shapes"?

3) Which parts of VT practices need to be further improved?

In order to better explore three research questions mentioned above, different research methods were applied accordingly (see the Table 3.2.1):

Research Questions	Research Methods
RQ1	Pre-test and post-test
RQ2	Lesson observation and document analysis
RQ3	Pre-test and post-test

Table 3.2.1: Research questions and Methods

3.3 Research object

The action research was conducted in one primary four class (a total of 31 students, including 14 boys and 17 girls) in the author's practicum school chosen by convenient sampling. It was a mixed-sex primary school located in Sha Tin District. Students in this school had relatively high learning abilities, and most students had a good foundation in mathematics learning.

3.4 Procedure

With reference to the structure of action research mentioned earlier, Table 3.4.1 shows the research steps and time.

Date	Research step
~Mid Feb., 2019	The author determined the research theme and research questions.
March 15th, 2019	The author tried to found out the situation at that time and identified students' state in *"Fitting shapes"* and the critical features through

	analyzing pre-test results.
March 18th ~ March 21st, 2019	The author used the initial analysis of step 2 to design the lessons. Referring to VT, the author provided various examples for students to discern the critical features of the object of learning, solving the students' confusion or difficulties, and strengthening students' spatial sense. A total of four lessons were spent on teaching all CFs, each lesson 35 minutes long. During the lessons, the author paid attention to the students' responses to the questions and whether the hands-on activities proceed smoothly, and recorded them timely (in the teaching process of the lessons, see Part 4.3.1 - 4.3.4) as evidence for later analysis.
March 22nd, 2019	The author evaluated the effects by conducting the post-test. After collecting the post-test results, the teacher compared the percentages of the options or the answers of the same question in the pre-test and post-test to see whether the students' understanding of "fitting shapes" enhanced or not. According to the comparison, she reviewed the teaching effectiveness and made a critical reflection based on the quantitative data as well as the observation evidence.
~May 10th, 2020	She revisited the original research question and assessed the extent of addressing it. Lastly, she shared results and wrote a thesis.

Table 3.4.1: Research steps and time

4. Applying Variation Theory in "fitting shapes"

4.1 Identify the object of learning from the unit to the topic

Since "fitting shapes" is a topic under the unit "fitting and dissecting shapes" （圖形拼砌與分割）, from the perspective of knowledge coherence, the author conducted a holistic analysis of "fitting shapes" and "dissecting shapes"（圖形分割）first. She summarized the following object of learning in this unit:

- Follow the specific instructions to form shapes or dissect shapes.
- See the diverse possibilities of fitting shapes and dissecting shapes.
- Strengthen understanding of the characteristics of each figure and the relationships between graphics.
- Cultivate spatial sense and problem-solving skills.

Among the objects of learning mentioned above, the most vital one was to identify the characteristics of all the graphics and clarify the relationship between figures. It served as the learning basis of both "fitting shapes" and "dissecting shapes". Since "fitting shapes" were taught before "dissecting shapes", the author tended to take the teaching of the former one, "fitting shapes", to demonstrate how to help students get command of this focal point.

4.2 Explore the critical features from the pre-test results

The author designed a diagnostic test paper (see Appendix 1) based on the learning content and required students to complete it before the lessons started. It helped the author ascertain their learning state based on the results of the pre-test and determine the critical features of the topic. Totally, there were eight questions in the pre-test. It consisted of several types of questions, including multiple-choice questions, fill-in-the-blank questions, and drawing questions. The author hoped to examine all the knowledge and abilities of shape and space that students had possessed, so the questions involved not only "fitting shapes" (a total of five questions, including Question 3,4,5,6,8) but also "dissecting shapes" (a total of four questions, including Question 1,2,7,8). All the features in this unit had been examined through this paper. The corresponding features that each question assessed are shown in the table below:

Question	Features
1	Assist dissecting shapes by analyzing the properties of rectangle and isosceles triangle.
2	Identification of the different graphics' characteristics that have been learned.
3	Different figures must have equal sides to fit together. / Identification of the trapezoid's characteristics.
4	Different figures must have equal sides to fit together. / Assist fitting shapes by analyzing the initial shapes and parallelogram.
5	Different figures must have equal sides to fit together. / Assist fitting shapes by analyzing the initial shapes and the potential answers.
6	Different figures must have equal sides to fit together. / Assist

	fitting shapes by analyzing the initial shapes and the potential answers.
7	Extraction of the area information of the figures on the graph paper （方格紙）. / Different ways of composing graphics of the same area.
8	Extraction of the area information of the figures on the graph paper. / Different ways of composing graphics of the same area.

Table 4.2.1: Corresponding features of each question

Chart 4.2.1 below shows the accuracy of each question:

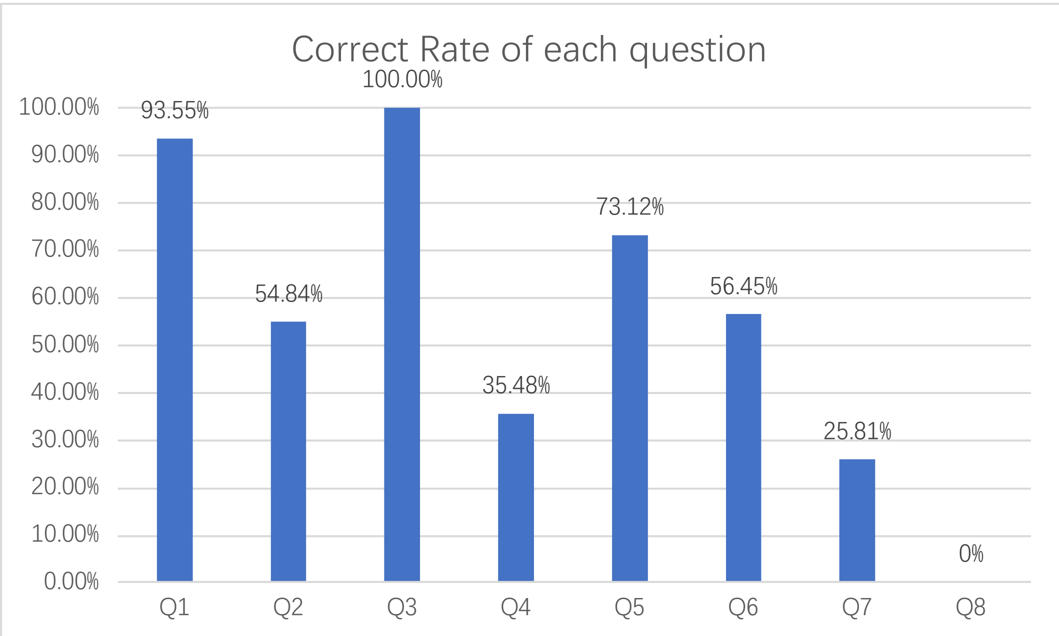


Chart 4.2.1: Correct Rate of each question in pre-test

The extremely high correct rate of Question 1 (93.55%) and Question 3 (100%) indicated that students had a preliminary understanding of the traditional and simple way of "fitting and dissecting shapes". As for Question 2, some students' problems arose from inadequate recognition or impression on the characteristics of the graphics, which was the students' previous knowledge in shape and space. Question 7 and Question 8 were relatively more focused on examining students' much higher problem-solving abilities in dissecting shapes. However, most of the students still have a lot of room for improvement in the basic skills of fitting shapes. Hence, the author only selected "fitting shapes" as the main research focus, hoping to find the critical features of it through further analysis of Question 4, Question 5, and Question 6.

4.2.1 Identify the critical features thorough further analysis

Question 4:

There were candidates for all four options (see the figure). It indicated that some students who do not understand chose their answers randomly. Multiple-choice questions could be related to luck. Some students might choose option D, which was the right answer, by chance. 48.39% of the class (15 students) chose option C as their answers. These students might judge by their feeling since III and IV were morphologically similar. The students noticed that both III and IV had a right-angled side, a hypothenuse, and upper（上底）and lower bottom edges（下底）. It seemed that they only need to flip and simply move IV to fit together with III, getting a parallelogram. However, they did not judge with the characteristics of the parallelogram: two groups of the opposite sides of a parallelogram are equal.

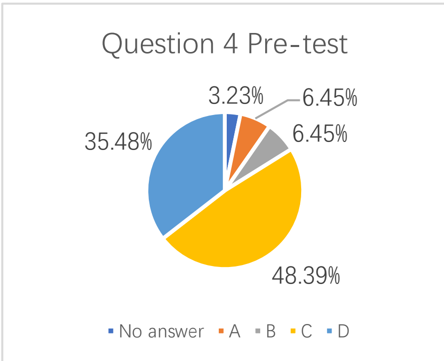


Figure 4.2.1_1: Q4 pre-test selection rate of each option

Question 5:

This question examined the possibilities of fitting shapes. 41.94% of the class (13 students) wrote all three correct answers. The percentage of answers for each figure was as follows: 74.19% of the class (23 students) wrote rectangle, 74.19% of the class (23 students) wrote trapezoid, and 70.97% of the class (22 students) wrote parallelogram. The percentages of answers for each graphics were very similar, and this showed that students similarly lacked in the mastery of rectangle, trapezoid and parallelogram. Therefore, more examples of various graphics should be included in the teaching process.

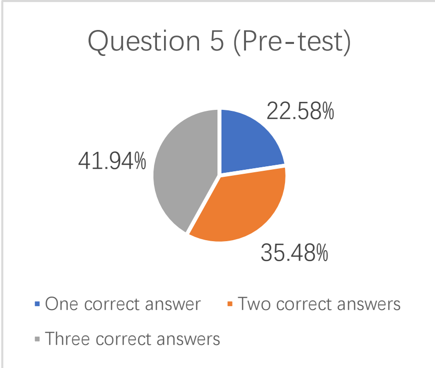


Figure 4.2.1_2: Q5 pre-test selection rate of each option

Question 6:

In this question, the number of graphics involved in fitting had increased from three to four. The types of graphics obtained from fitting also increased. Besides, the built-square was not placed in a standard way. Students had to fit the small pieces together first and then tried to distinguish the square. Therefore, the difficulty level of Question 6 was increased compared to Question 5. The test results reflect that students were not sufficiently familiar with

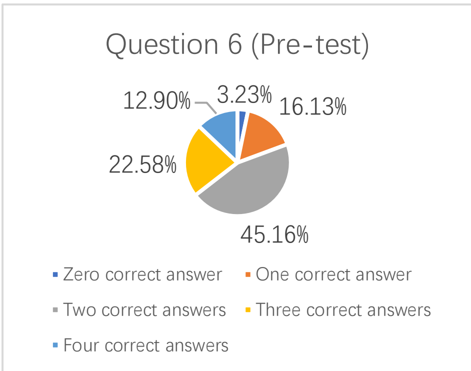


Figure 4.2.1_3: Q6 pre-test selection rate of each option

squares, trapezoids, and parallelograms. These graphics should be more involved in teaching.

Through the above analysis, the related critical aspects for discernment in this class were specified. The *critical features* (CF) identified were:

CF1. Different figures must have equal sides to fit together.

Two kinds of methods could be used to determine whether the lengths of the two sides are the same:

1. Overlap of two arbitrary edges from different graphics: Students may use imagination or practice to transform the graphics and determine whether two sides can overlap. If two independent edges from two different shapes can overlap, these two shapes can be fitted together. There are mainly three kinds of graphics transformation, including rotation, translation, and flip over. But it should be noted that students do not need to master the specific mathematical terms of these three methods. They only need to know how to transform the graphics roughly.
2. Observation of the length of each side of all graphics: Extract the length information of the corresponding figures from the graph paper (grid sheet) or the pin point paper to determine which two sides are equal in length. Two equal-length edges from two figures can be tried to fit together.

CF2. Fit shapes by analyzing the properties of the original graphics and the obtained graphics.

The properties of triangles and quadrilaterals that students learned before are summarized in Table 4.2.1_1 – 4.2.1_2:

Type of triangles	Properties
An equilateral triangle	has three equal sides.
An isosceles triangle	has two equal sides.
A scalene triangle	has no equal side.
A right-angled triangle	has a right angle.
An isosceles right-angled triangle	has two equal sides and a right angle.

Table 4.2.1_1: Properties of triangles

Quadrilaterals / Properties	**Square**	**Rectangle**	**Parallelogram**	**Rhombus**	**Trapezoid**
Opposite sides are parallel.	√	√	√	√	
Only one group of opposite sides is parallel.					√
Opposite sides are congruent.	√	√	√	√	
Four sides are equal.	√			√	
Opposite angles are congruent.	√	√	√	√	
Four angles are equal.	√	√			

Table 4.2.1_2: Properties of quadrilaterals

There are relationships between these graphics (see Table 4.2.1_3 and Table 4.2.1_4).

For example, two identical small triangles can be pieced together to form a quadrilateral or a large triangle.

Two	(scalene) triangles	of exactly the same size and shape can make...	a parallelogram.
	isosceles triangles		a parallelogram/ a rhombus.
	equilateral triangles		a rhombus.
	right-angled triangles		a parallelogram/ a rectangle/ an isosceles triangle.
	isosceles right-angled triangles		a parallelogram/ a square /a big isosceles right-angled triangle.

Table 4.2.1_3: Relationship between the graphics (1)

Two identical small quadrilaterals can be pieced together to form a large quadrilateral.

Two	squares	of exactly the same size and shape can make...	a rectangle.
	rectangles		a big rectangle.
	parallelograms		a big parallelogram.
	rhombuses		a parallelogram.
	trapezoids		a parallelogram.

Table 4.2.1_4: Relationship between the graphics (2)

At this learning stage, most questions that the students are faced with are what graphics can be made by using two or more graphics. By comparing the properties of original graphics and the obtained graphics, students might find some clues to solve the problem. For instance, when students get two identical isosceles right-angled triangles for fitting shapes, except the parallelogram, they should be able to consider the possibilities of square and rectangle since two isosceles right-angled have already provided two right angles. Among the quadrilaterals that students have learned, only square and rectangle have more than two right angles in the quadrilaterals.

4.3 Designing patterns of variation for teaching critical features

Aiming at letting students discern the critical features, the author used VT to design the lessons with various examples and activities. The overall arrangement for teaching critical features in the four lessons has been summarized in Table 4.3. Part 4.3.1 – 4.3.4 mention how to carry out the activities in the classroom from Lesson 1 to Lesson 4 and introduce the corresponding patterns of variation that have been applied.

	Arrangement for teaching critical features
Lesson 1	Understand CF1 and apply the first method (Overlap of two arbitrary edges from different graphics) to determine whether the lengths of two sides are the same.
Lesson 2	Understand CF1 and apply the first method to determine whether the lengths of two sides are the same; Roughly know about CF2.
Lesson 3	Understand CF1 and applying the second method (Observation of

	the length of each side of all graphics) to determine whether the lengths of two sides are the same; Understand CF2 and know how it helps solve the problem of "fitting shapes".
Lesson 4	Same as Lesson 3; Analyze the comprehensive situation of "fitting shapes" with the understanding of CF1 and CF2.

Table 4.3: Overall arrangement for teaching critical features

4.3.1 Teaching process of Lesson 1 (L1)

In the first lesson, the teacher began her teaching by leading the students to review the properties of the shapes (both triangles and quadrilaterals, as shown in Table 4.2.1_1 and Table 4.2.1_2) they learned before. Then she asked students to take out the *Tangram* （七巧板） that they had prepared, intending to stimulate their interest in fitting shapes. The tangram consists of seven pieces, including five isosceles right triangles of different sizes, a square, and a parallelogram. The teacher chose tangram as the teaching aid because the students had already used it when they studied <Shape and Space> in primary 2, and they had mastered the properties and characteristics of all these graphics in the third and fourth grades of learning. The teaching of fitting shapes got started from exploring the relationship between the boards and boards.

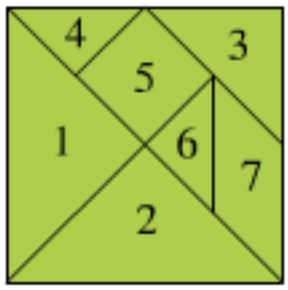


Figure 4.3.1: Tangram

To make students discern CF1, the teacher first gave an example, requiring the students to use No. 4 and No. 6 boards in the tangram to make No.3/ No.5/ No.7 respectively. When explaining and displaying the dynamic transformation process of how to use No.4 and No.6 boards to build No.3/No.5/No.7 boards, the teacher asked students that "Can these two edges fit together?" "If yes, what graphics will we get?" The teacher demonstrated the counterexample of fitting shapes, joining together the two edges of unequal length from two graphics and asked the students "Can these two edges fit together?". Then the students found out that they could not even get regular shapes (e.g. triangles, quadrilaterals at this learning stage) if they construct graphics in this way.

In addition to questioning and interacting with students, she also allows students to gain hands-on experience. Each student had a set of tangrams. Under the guidance of the teacher, they got the No.4 and No.6 boards in tangram, moving and stitching different edges to get different graphics (No.3/No.5/No.7). Students then found that, to form a regular shape, two equal-length sides from two figures must be fitted together. It caused a heated discussion among the students since they surprisingly discovered that they were able to construct more graphics than they think with only two shapes. Two congruent graphics have multiple sets of sides of equal length to fit together so that diverse figures may be gained. Through this process, students directly realized that overlap of two arbitrary edges from different graphics, which is one method of finding sides that meet the requirement of CF1.

Invariant	Varied	What is to be discerned
Use No.4 and No.6 boards in the tangram	To respectively make No.3/ No.5/ No.7 boards	CF1 - Overlap of two arbitrary edges from different graphics No.4 and No.6 boards have two groups of equal sides. Combining the equal sides respectively, three different figures can be obtained.

Table 4.3.1_1: Pattern of Variation in L1 (Contrast)

Then the teacher prepared two more examples for students to consolidate such a method of finding two sides with the same length for fitting shapes. Students were required to use three different groups of boards to make No.1 board in the tangram and use the same group of boards to make a triangle, a rectangle and a parallelogram respectively.

Invariant	Varied	What is to be discerned
Adopt the same method in fitting	Use No.4, No.6, No.3 boards/ No.4, No.6, No.5 boards/ No.4, No.6, No.7 boards in the tangram to make No.1 board in the tangram	CF1 – Overlap of two arbitrary edges from different

shapes		graphics
	Use No.4, No.6 and No.7 boards in the tangram to respectively make a triangle/ a rectangle/ a parallelogram	

Table 4.3.1_2: Pattern of Variation in L1 (Generalization)

The pattern of variation used in the above teaching process of L1 is contrast and generalization (see Table 4.3.1_1 and 4.3.1_2). In the lesson, students used the same group of graphics (No.4 and No.6) to form different shapes. Since the essence of forming different shapes was to put different groups of equal-length together, the students realized that some edges could be fitted together while others could not. This could be seen as using contrast. In addition to the first example, the teacher also used two more examples to let students know that the first method of CF1 can be generalized and applied to all these examples in "fitting shapes".

The progress of students was visible and gratifying. When students tried to move the pieces of tangram and form new graphics, they easily ignored the reversal (flip over) so that they did not smoothly find out how to form some specific shapes. To deal with this situation, the teacher also made a generalization through several examples, prompting students to remember to think about the possibility of reversal of each graphics in fitting shapes. After the teacher displayed how to get the final graphics from the initial state through several dynamic transformations and encouraged the students to imitate the transformation process through hands-on practices, the students established a preliminary impression of the graphics transformation. Students' ability of fitting shapes had been improved as time went by. For example, they needed to refer to No.3 (placed No.3 on their desk) and moved No.4 and No.6 to achieve the goal in the beginning. But later, students could use three boards to make some specified figures without any reference objects. For example, they used No.4, No. 6, and No.7 to make a triangle, a rectangle, or a parallelogram respectively.

4.3.2 Teaching process of Lesson 2 (L2)

In the second lesson, the teacher no longer relied on the tangram, but used the common plane figures for teaching "fitting shapes". Similar to the first lesson, the teacher also used questioning and hands-on experience to let students discern the CF1. In order to make different quadrilaterals listed in Table 4.3.2_1, the students had to overlap two arbitrary edges from the small graphics until they found out the edges with the same length and put them together.

Invariant	Varied	What is to be discerned
Adopt the same method in fitting	Use two congruent right-angled triangles to make two different quadrilaterals.	CF1 – Overlap of two two

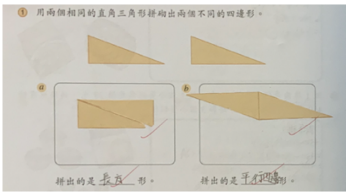
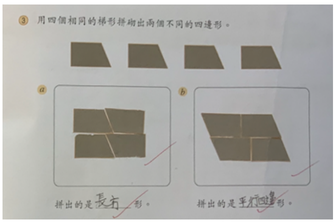
shapes		arbitrary edges from different graphics
	Use four congruent trapezoids to make two different quadrilaterals. 	

Table 4.3.2_1: Pattern of Variation in L2

In the latter part of L2, the examples prepared by the teacher shifted from figure-based to text-based. This required students to gradually replace physical assistance with imagining the process of fitting shapes. Some students with weak learning level were not able to get the correct answer, the teacher then took out the prepared teaching aids (the paper-cut of the specific graphics) and demonstrated the process of fitting shapes to help them solve the problem. Although these two examples were still mainly used to evaluate the students’ mastery of CF1, at the end of the lesson, the teacher proposed another way to solve the problem was to think backwards, thus introducing CF2. Starting from the properties of the graphics in the options and connecting with the characteristics of the original graphics, students found that they could piece the edges together in a purposeful way.

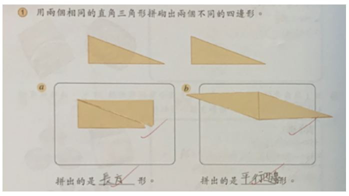
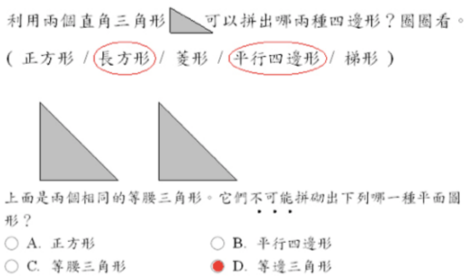
<table>
<tr><th>Invariant</th><th>Varied</th><th>What is to be discerned</th></tr>
<tr><td rowspan="2">Fitting shapes with common plane figures</td><td>Figure-based questions

</td><td rowspan="2">CF1 – Overlap of two arbitrary edges from different graphics;</td></tr>
<tr><td>Text-based questions

</td></tr>
</table>

Table 4.3.2_2: Pattern of Variation in L2

4.3.3 Teaching process of Lesson 3 (L3)

The third lesson focused on learning "fitting shapes" on the pin point paper（釘點紙）, which is a useful auxiliary tool for fitting shapes. Students could get rid of guessing about whether the sides are equal, but have specific values as a reference. Table 4.3.3 demonstrated the first designed activity and the corresponding patterns of variation used in the lesson.

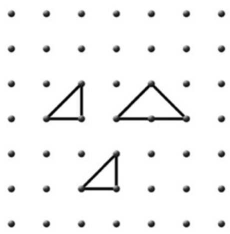
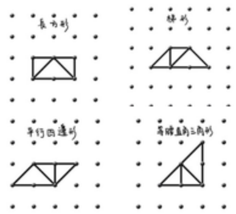
Invariant	Varied	What is to be discerned
Use the same group of graphics on the pin point paper 	To respectively make a rectangle/ a trapezium/ a parallelogram/ a isosceles triangle 	CF1 – Observation of the length of each side of all graphics; CF2

Table 4.3.3: Pattern of Variation in L3

Facing the new types of situations, teachers used the questioning method to guide students' learning. After the students tried to solve the problem and draw their answers on the worksheets, the teacher asked, "Are you sure that these sides correspond and the small triangles are as big as the given one?" "Our goal is to make a rectangle. What kind of characteristics or properties does a rectangle possess?" "Could you find the right angles in the original graphics? How can we make more right angles by using these graphics?" Some of the students began to aware that the characteristics of the quadrilaterals could also help them solve problems. The teacher then led the students to observe the number of grids occupied by each side of the triangles. After that, the students were required to consider "Each of the two small triangles has a right angle. The rectangle that we would like to make has four right angles. If we do not move the large triangle, how can we transform the small triangles and make the other two right angles with the large one?" Applying the knowledge of CF1, students considered joining two sides of the same length together, and they finally found that two new right angles were formed.

At first, some students believed that the pinpoint paper was just another plane for fitting shapes, which was no different from fitting shapes on a paper with a blank background. But through these experiences, they started to pay attention to the length information provided by the pinpoint paper. Combined with the characteristics of the quadrilaterals they have learned before, they were able to check whether the graphics they built are reasonable. What's more, they came to realize that they could consider putting two edges of equal length together to look for some clues to solve the problem even if they have no idea about how to fit shapes.

Students were given chances to have their hands-on experiences too. They made three triangles through using the rubber bands of three different colors on the electronic nail board（電子釘板）. And they were allowed to see the graphics transformation process on their iPad, which could be seen as a visual stimulus of fitting shapes to them. The teacher observed that all the students have tried to overlap two or more groups of edges with equal length, thereby they all made progress in the diversity of fitting shapes.

4.3.4 Teaching process of Lesson 4 (L4)

In Lesson 4, the plane used for fitting shapes changed from the pinpoint paper to the graph paper. The point on the pin point paper is the intersection of grid lines on the graph paper. Therefore, the processes of solving "fitting shapes" questions on these two planes were quite similar. The only difference was that students had to learn to extract the length information of the edges from the graph paper. The teacher prepared the following task for the students, and the pattern of variation is shown in Table 4.3.4_1.

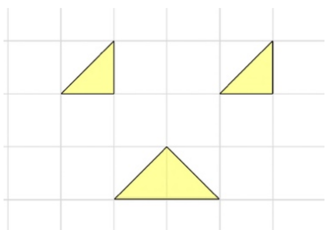
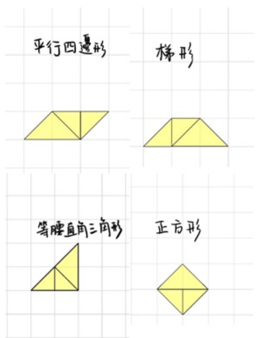
Invariant	Varied	What is to be discerned
Use the same group of graphics on the graph paper 	To respectively make a parallelogram/ a trapezium/ a isosceles triangle/ a square 	CF1 – Observation of the length of each side of all graphics; CF2

Table 4.3.4_1: Pattern of Variation in L4

With the previous experience of fitting shapes on the pin point paper, most students completed this task quickly and with high accuracy. When the teacher randomly selected students to express their ideas, all of them first stated the length information of the original graphics and the properties of the final one. Some students even noticed and told the teacher that the initial group of the graphics in this task were exactly the same (sizes of the graphics, and the number of the graphics) as those in the first task of L3. The students then found that the background was not a factor affect fitting shapes, but an auxiliary tool for solving "fitting shapes" questions.

Invariant	Varied	What is to be discerned
Using the same group of the graphics in fitting shapes	On different backgrounds (pin point paper and graph paper)	CF1 – Observation of the length of each side of all graphics

Table 4.3.4_2: Pattern of variation in L3/L4 (Separation)

Towards the end of the teaching of this topic, students' familiarity with fitting shapes were relatively higher than before, so the difficulty of the questions and exercises had been increased. Students no longer need to use the option as a reference to try out "fitting shapes", but can obtain the information of the graphics from their observation, and know how to stitch the edges to get as many possibilities as possible.

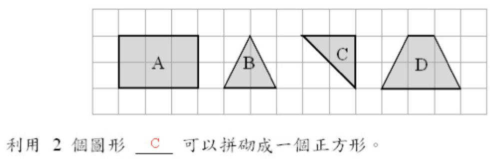


Figure 4.3.4_1: Examples of variation in L4

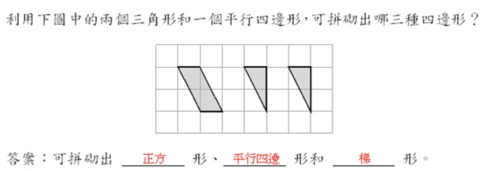


Figure 4.3.4_2: Examples of variation in L4

Invariant	Varied	What is to be discerned
Questions under the same topic of "fitting shapes"	Difficulty (e.g. more types of graphics involved in fitting shapes); the forms of the questions (e.g. from the picture-based questions to text-based questions)	CF1 and CF2

Table 4.3.4_3: Pattern of variation in L4 (Fusion)

5. Findings

Chart 5.1 shows the comparison of the overall situation of the pre-test and post-test. Chart 5.2 to chart 5.5 specifically compare students' answers of Q5 and Q6 in the pre-test and post-test.

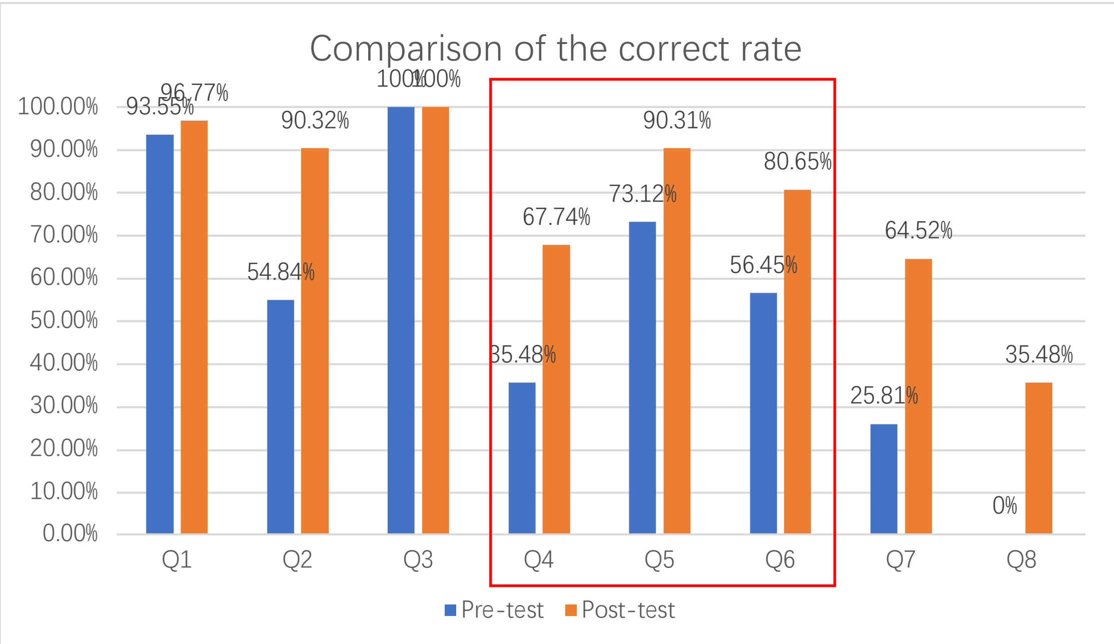
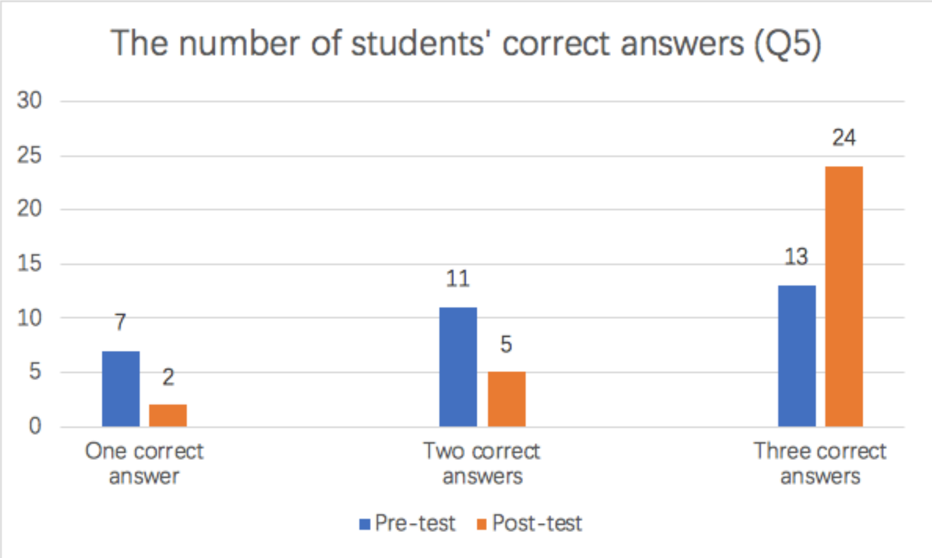


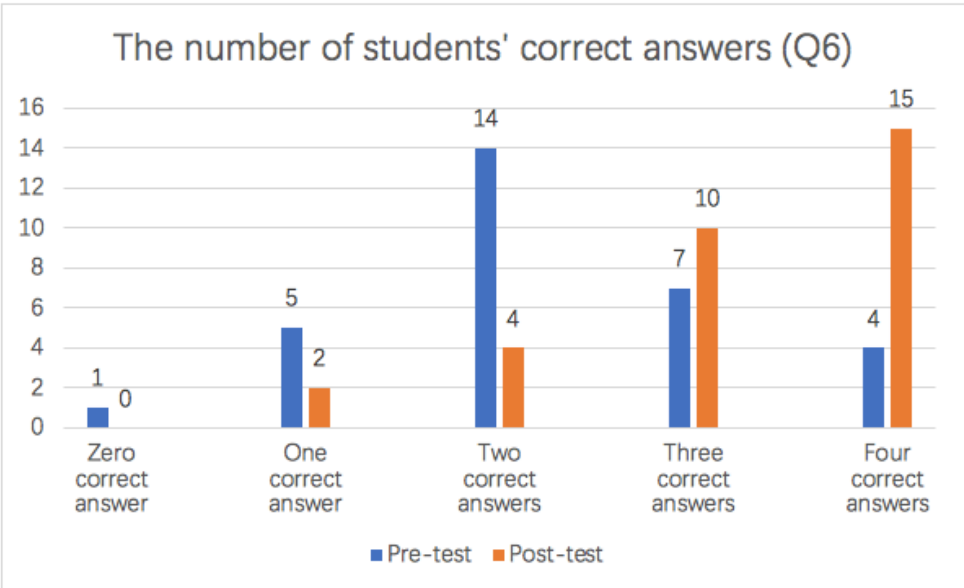
Chart 5.1: Comparison of the correct Rate of each question in post-test and post-test



Pre-test Average: 2.1936 correct answers

Post-test Average: 2.7097 correct answers

Chart 5.2: Comparison of the number of Q5 correct answers of students between pre-test and post-test



Pre-test Average: 2.2579 correct answers

Post-test Average: 3.2259 correct answers

Chart 5.3: Comparison of the number of Q6 correct answers of students between pre-test and post-test

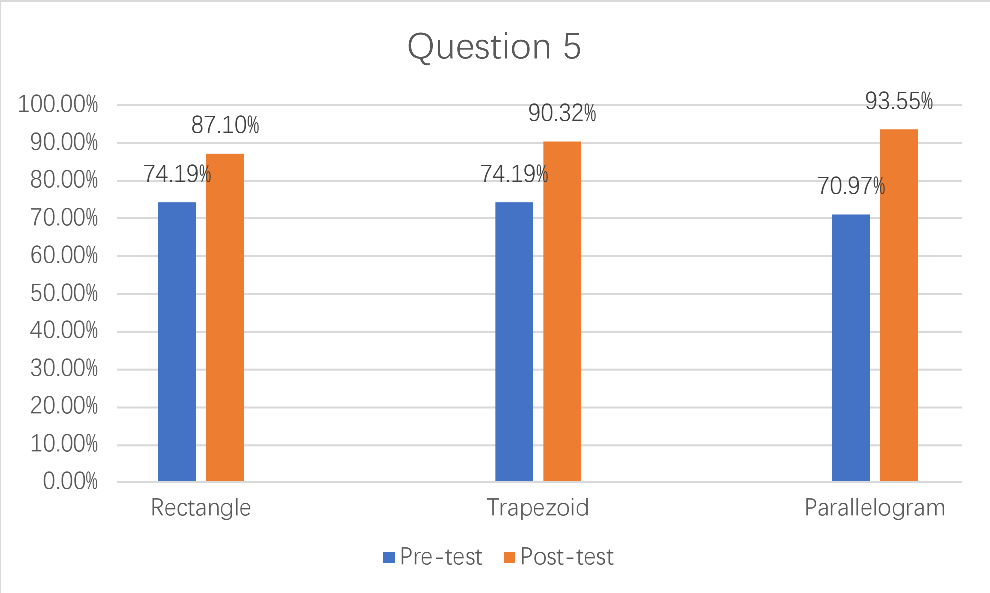


Chart 5.4: Percentage comparison of students who answered each graphics between pre-test and post-test of Q5

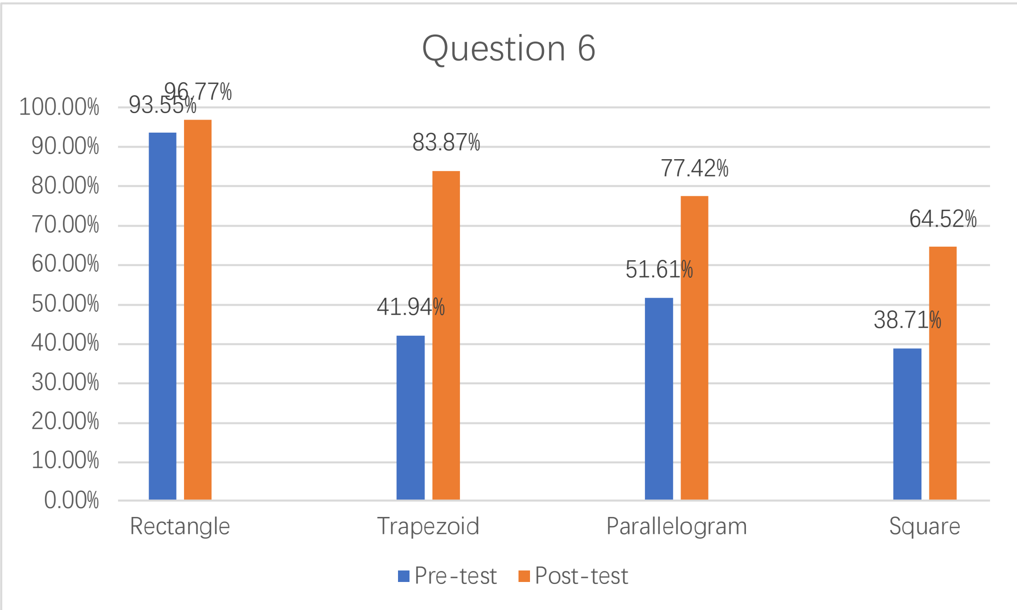


Chart 5.5: Percentage comparison of students who answered each graphics between pre-test and post-test of Q6

According to the data shown in the tables above, the findings were organized as follows:

5.1 Impact of the use of VT on students' learning of "fitting shapes"

Overall, the correct rates of Q4, Q5 and Q6 increased by 32.26%, 17.19% and 24.2% respectively. This showed that using VT productively enhanced students' learning of "fitting shapes" to some extent. Seen from the tables above, the types of figures that students built increased, and most of the students were able to find out all the possibilities of "fitting shapes" (the correct rate of Q5 and Q6 have reached 90.31% and 80.65% respectively). Students' familiarities to rectangle, trapezoid, parallelogram, and square were all relatively higher than that before learning. Even though the students did not know the specific concepts and professional mathematics language of the transformation methods, they knew how to shift the shapes roughly and get the answers.

Nevertheless, the accuracy of Q4, which only reached 67.74% in the post-test, still had a lot of room for improvement. This reflected that students were still not accustomed to fitting shapes by analyzing the length information (numerical value) of both the initial graphics and the obtained graphics, which can be improved by refining CF2 in the next lessons.

5.2 VT practices enhanced students' learning of "fitting shapes"

Looking back at the overall teaching, it was effective for teachers to use patterns of VT in the four lessons. To make students realize that using the same group of graphs can make different figures, 'contrast' was first used in the attempt of piecing different groups of edges together; 'generalization' was used in exploring the same methods of finding the equal edges; 'separation' was used in "fitting shapes" with the same groups of graphics in different backgrounds (pin point paper and graph paper), and 'fusion' was finally used in dealing with comprehensive situations of "fitting shapes" methodically. These four kinds of variation interaction acted together and reinforced each other in the lessons to let students discern the critical features and master the problem-solving skills, which sustained the pedagogical model of VT put forward by Leung (2012).

The teaching process also showed that systematically presenting the series of examples to students was indeed beneficial to their learning, which consolidating Watson and Mason's (2016) statement. Through the varied examples and hands-on activities based on the tangram in the first lesson and the common graphics in the second lesson provided by the teachers, students became much clearer about CF1 and they grasped the first method of finding the equal edges. In the third and fourth lesson, the variation of examples mainly displayed in the transition of the difficulty of the examples (e.g. varied backgrounds for fitting shapes, more types of graphics involved in fitting shapes, more types of graphics finally built), as well as the form of the questions (from the picture-based questions to the text-based questions). After this process, the students got command of the second method of finding the equal edges in CF1 and discerned CF2, which was significant in fitting shapes. Furthermore, the instructive questioning, the instant response to the students' views that the teacher interspersed in the four lessons allowed the students experience more variations in understanding of this topic, which supported the Lo's (2012) light on effective teaching strategies.

5.3 The practices of VT need to be further improved

However, compared to CF1, the teaching of CF2 seemed less successful. 60% (six of ten) students who made the wrong choice in the pre-test still chose C as their answers in Q4. They failed to think backwards and solve problems by taking advantage of the characteristic of parallelograms. This result indicated that the teacher had not applied enough counterexamples for fitting shapes in teaching, and neglected teaching students how to link the solution with the previous knowledge when facing this type of problem.

Besides, it is worth noting that 35.48% of the class (11 students) still did not give "square" as their answer in Q6, which led to the lower correct rate than Q5. During teaching, most of the squares that students were exposed to were drawn on a horizontal orientation. As a result, they did not think about this 'special' case when they tried to find out the answers. This showed that when the teacher arranged her teaching based on VT, she did not consider whether the examples involved were comprehensive enough (especially for the case of "fitting shapes" on the pin point paper or the graph paper), and ignored the methods of finding out all the possibilities without omission.

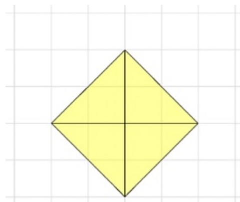


Figure 5.3.1:
'special' case of square

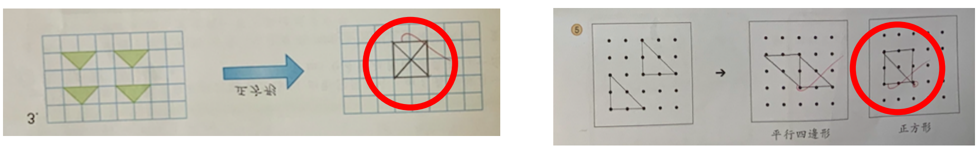


Figure 5.3.2/ 5.3.3: Most of the square that students saw in the lessons or exercises

6. Reflection

6.1 Research restriction

Firstly, the research object was limited to a small size. The research was originally expected to bring out more obvious results by setting the experimental group and control group in two different classes. However, the author was only assigned one fourth-grade class in her practicum. The other fourth-grade classes were taught by other teachers and the teaching schedule did not match. Hence, this research was unable to collect another group of data for analysis. Future research is expected to conduct in more classes and involve more students to prove the wide applicability of VT.

Secondly, the research methods of RQ2 were not enough to thoroughly support which parts of VT practices can enhance students' learning of "fitting shapes". In addition to observing the students' responses or their performance in the activities, the teacher should also conduct the pre-lesson and post-lesson interviews with students so as to ascertain students' mastery in various aspects of "fitting shapes". For example, let them talk about the methods that they have learned or the impressive activities that promote their understanding in the lessons.

6.2 Recommendations on improving the action research

6.2.1 The design and duration of pre-test and post-test

When the author designed the pre-test and post-test paper, considering that the mathematics level of students in this class was in the upper-middle-range, some of the questions were set comparatively difficult, exceeding the basic assessment requirements of the curriculum. Although it can cultivate problem-solving skills and higher-level thinking of students, it might be a burden for students with relatively weak mathematical abilities. Teaching itself should take care of student learning differences and assume that most students master the basic learning contents. It might be better to choose questions with moderate difficulty in the pre-test and post-test paper.

Additionally, the characteristics of the figures play important roles in this chapter. It is necessary to assess students’ discernment of these characteristics in the tests. For instance, a table can be added before all the questions. Let students fill in the properties of the triangles and quadrilaterals they have learned before in order to check students’ mastery level of the existing knowledge. At the same time, this table can also be used as a reference to assist students in completing other test questions. Students need to make full use of the properties of these graphics when solving problems.

What’s more, the type of questions may affect the validity of the test. Multiple choice questions could be replaced by the blank or short questions. Fill-in-the-blank questions and short questions can better show the students’ thinking process and results, making teachers more explicit about students’ perplexities and difficulties and helping teachers find out the critical features accordingly. For example, Question 4 can be changed to a fill-in-the-blank question, requiring students to write down the two graphics they chose.

Due to limited teaching time, both pre-test and post-test took only about 10 minutes. After the tests, some of the students mentioned that if they were allowed more time to think, they might work out more questions. Take a look at the whole test paper, the amount of questions is not too much for the teaching of a unit, and the forms of the investigation were quite diversified, including the multiple choices, fill-in-the-blanks, short questions without complicated written problem-solving process. But each students' mathematics ability is different and unique, and the speed of solving mathematics problems will be different. If some of them are not given enough time for finishing their test, it may not fully reflect the students' learning effectiveness. Thus, future researchers should reserve sufficient time for conducting the pre-test and post-test.

6.2.2 Engaging students with cooperative learning

In the lessons, the teacher provided a lot of hands-on opportunities for students to discern the critical features, but not all the students can actively and consciously participate in the classroom. Hence, their learning effectiveness was relatively lower than others. To cope with this, the teacher could find some way to organize the activities more systematically. Think Pare Share, a cooperative learning strategy, not only fosters students' engagement in their own learning (Kaddoura, 2013) but also motivates students to share their thoughts with their peers to gain further knowledge (Robertson, 2006). It can be considered to be applied in classroom teaching.

Take an activity of discerning CF1 as an example. At first, the students should try to overlap different groups of two edges and establish their own ideas, knowing the circumstances under which different shapes could be fitted together. Then, students could be grouped in pairs and share their thoughts. In this step, students can teach others, solve their doubts and reach the consensus. In this way, students can play roles of problem-solvers as well as the observers, which helps them deepening the impression of fitting shapes. Finally, the students may share their concrete consideration process of fitting shapes with a larger group. Presentations help them to organize their thoughts better and cultivate their mathematical expression skills. In addition to this collaborative learning process, teachers should provide more consolidation questions for students to apply their own skills, allowing them to complete the integration and the 'generalization' of the knowledge.

6.2.3 Deepening understanding of abstract tasks with electronic teaching and useful apps for teaching mathematics

During the lessons, the author found that students can better understand the principles of fitting and dissecting shapes by using teaching aids. At this age, it is difficult for students to imagine how graphics transform and finally piece together. Most of the P.4 students are still in the stage of concrete operational (Piaget, 1977). Only by providing as many opportunities as possible for students to practice, the students may build up a spatial sense. Using visual teaching aids is essential for students when they initially learn about knowledge about shape and space. However, the practicality of some electronic teaching aids has also been proved in daily teaching.

Generally speaking, when a teacher teaches units related to shape and space, they need to spend a lot of time to prepare for the teaching aids. It is not environmentally friendly since most of the teaching aids are just designed for one specific question or activity. In recent years, the government has actively promoted and introduced electronic teaching aids in the classroom. Thanks to this practicum in an elementary school that promotes electronic teaching, the author got to know plenty of apps for teaching mathematics, such as graphic activity apps from Modern Education Research Society and Geoboard. The author deems that these apps will enter into more math classes overtime to assist teaching.

6.3 Recommendations on the application of VT in "fitting shapes"

First of all, teaching contents can be adjusted appropriately. The author initially taught "fitting shapes" and "dissecting shapes" separately. However, what students are expected to learn in this topic, or in this unit, is to be clearer about the interactive relationship among the graphics. The interaction of the graphics can take place both in "fitting shapes" and "dissecting shapes". For example, two identical isosceles right angles can be made into a square, but in another way, a square can be divided into two equal isosceles right triangles. Putting these two processes together and comparing them may help students deepen their impressions on the relationship of the graphics.

In addition, some students may not be able to completely handle the abstraction process of fitting shapes because they have not reached to a specific developmental stage, which is consistent with the worries of Hanfstingl, Benke and Zhang (2019). Even if the students experienced the variation, they still had difficulty in solving the abstract situations. To achieve better teaching effect, teachers should pay attention to the learning barriers of the lower level students and provide support to help them get command of the CFs, for example, allowing these students more time to use learning aids (e.g. the paper cut of the graphics) for a more extended period.

7. Conclusion

Under the background of curriculum reform, this action research attempted to put the VT into practice, as well as tested its utility and feasibility in primary mathematics teaching. Taking "fitting shapes" as the teaching topic, this action research was conducted in the context of the Hong Kong local P.4 classroom. Referring to the action research model and the basic principles of VT, the author first analyzed the learning state of the students, and then designed and completed the teaching of two CFs. Looking at the quantitative data, students' overall understanding of this topic enhanced. From the teacher's observation in class, the teaching process with the application of four variation patterns, the series of examples, and appropriate teaching interaction facilitated students' learning. Nevertheless, there were still many deficiencies in the research that need to be improved, such as the diversity of examples, the design of the pre-test and post-test papers, or the organization of the teaching contents. These need to be further improved in further research or classroom Learning Study. Through this research, the author has more affirmed the supporting role of VT in class design and example selection in daily mathematics teaching. For a fledgling teacher, this theory will greatly benefit her future teaching and professional development.

Reference

Bowden, J., & Marton, F. (1998). *The University of learning—beyond quality and competence in higher education*. London: Kogan Page.

Bussey, T., Orgill, M., & Crippen, K. (2013). Variation theory: A theory of learning and a useful theoretical framework for chemical education research. *Chemistry Education Research and Practice*, *14*(1), 9-22.

Cheng, E. (2016). Learning through the Variation Theory: A Case Study. *International Journal of Teaching and Learning in Higher Education*, *28*(2), 283-292.

Cheong Cheng, Y. (2009). Hong Kong educational reforms in the last decade: Reform syndrome and new developments. *International Journal of Educational Management*, *23*(1), 65-86.

Chiu, L. T., & Bo, W. L. (2012). Teaching Cantonese opera in a primary school. *International Journal for Lesson and Learning Studies*, *1*(3), 261-275.

Curriculum Development Council, (2014). *Basic Education Curriculum Guide – To Sustain, Deepen and Focus on Learning to Learn (Primary 1 – 6)*. Hong Kong: Curriculum Development Council.

Driver, A., Elliott, K., & Wilson, A. (2015). Variation theory-based approaches to teaching subject-specific vocabulary within differing practical subjects. *International Journal for Lesson and Learning Studies*, *4*(1), 72-90.

Education Bureau (2017). Supplement to Mathematics Education - Key Learning Area Curriculum Guide. Retrieved from https://www.edb.gov.hk/attachment/en/curriculum-development/kla/ma/curr/pmc2017_e.pdf

Hanfstingl, B., Benke, G., & Zhang, Y. (2019). Comparing variation theory with Piaget's theory of cognitive development: More similarities than differences? *Educational Action Research: Lesson Study*, *27*(4), 511-526.

Hella, E., & Wright, A. (2009). Learning 'about'and 'from'religion: Phenomenography, the variation theory of learning and religious education in Finland and the UK. *British Journal of Religious Education*, *31*(1), 53-64.

Kaddoura, M. (2013). Think Pair Share: A Teaching Learning Strategy to Enhance Students' Critical Thinking. *Educational Research Quarterly*, *36*(4), 3-24.

Kullberg, A., Runesson, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM - the*

International Journal on Mathematics Education, *49*(4), 559-569.

Lam, Suk Wah Louisa. (2002). *Mathematics Education Reform in Hong Kong*. Retrieved from http://dipmat.math.unipa.it/~grim/SiLam.PDF

Lam, H. C., & Tsui, B. M. A. (2013). Drawing on the variation theory to enhance students' learning of Chinese characters. *Instructional Science*, *41*(5), 955-974.

Leung, A. (2012). *Variation and Mathematics Pedagogy*. Place of publication not identified: Distributed by ERIC Clearinghouse.

Lo, M. (2012). *Variation theory and the improvement of teaching and learning* (Gothenburg studies in educational sciences; 323). Göteborgs, Sweden]: Acta Universitatis Gothoburgensis: Göteborgs Universitet.

Luhmann, N. (1990). The cognitive program of constructivism and a reality that remains unknown. In W. Krohn, G. Küppers, & H. Nowotny (Eds.), *Selforganization*. *Portrait of a scientific revolution* (pp. 64–85). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Marton, F. (1986). Phenomenography: A research approach to investigating different understandings of reality. *Journal of Thought*, *21*(3), 28-49.

Marton, F., & Booth, S. (1997). *Learning and Awareness*. Mahwah, NJ: Lawrence Erlbaum.

Marton, F., & Pang, M. (2007). *Connecting student learning and classroom teaching through the variation framework*. Paper presented at the 12th Conference of the European Association for Research on Learning and Instruction, Budapest, Hungary. Retrieved from http://www.economicsnetwork.ac.uk/showcase/marton_variation.pdf.

Marton, F., & Tsui, A. (2004). *Classroom discourse and the space of learning.* Mahwah, N.J.: Lawrence Erlbaum Associates, Pub.

McAteer, M. (2013). *Action research in education*. London: SAGE Publications.

Mills, G. (2007). *Action research: A guide for the teacher researcher*. 2. ed. Upper Saddle River, NJ: Pearson Merrill Prentice Hall.

Parsons, R., & Brown, K. (2002). *Teacher as reflective practitioner and action researcher*. Belmont, CA u.a.: Wadsworth.

Rittle-Johnson, B., Star, J., & Durkin, K. (2009). The Importance of Prior Knowledge

When Comparing Examples: Influences on Conceptual and Procedural Knowledge of Equation Solving. *Journal of Educational Psychology*, *101*(4), 836-852.

Robertson, K. (2006). *Increase student interaction with "ThinkPair-Shares" and "Circle Chats"*. Colorin: Colorado.

Rundgren, C., & Tibell, L. (2010). CRITICAL FEATURES OF VISUALIZATIONS OF TRANSPORT THROUGH THE CELL MEMBRANE—AN EMPIRICAL STUDY OF UPPER SECONDARY AND TERTIARY STUDENTS' MEANING-MAKING OF A STILL IMAGE AND AN ANIMATION. *International Journal of Science and Mathematics Education*, *8*(2), 223-246.

Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge journal of education*, *35*(1), 69-87.

Schwartz, D., & Bransford, J. (1998). A Time For Telling. *Cognition and Instruction*, *16*(4), 475-5223.

Stringer, E. T. (2014). *Action research*. Los Angeles, CA: SAGE.

Ting, J., Tarmizi, R., Bakar, K., & Aralas, D. (2017). The Adoption of Variation Theory in the Classroom: Effect on Students' Algebraic Achievement and Motivation to Learn. *Electronic Journal of Research in Educational Psychology*, *15*(2), 307-325.

Tse, S. K., Marton, F., Ki, W. W., & Loh, E. K. Y. (2007). An integrative perceptual approach for teaching Chinese characters. *Instructional Science*, *35*(5), 375-406.

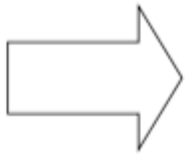
Watson, A., & Mason, J. (2006). Seeing an Exercise as a Single Mathematical Object: Using Variation to Structure Sense-Making. *Mathematical Thinking and Learning*, *8*(2), 91-111.

梁玉麟、勞傅燕華、江巧妍、李華昌、鄭智強、梁志強、黃德華與黎耀志 (2011)。*《數學課堂學習研究實踐與數學基本概念的教學 (第 1 版. ed., 香港課堂學習研究叢書)》*。合肥市: 安徽教育出版社。

Appendix (Pre-test paper/ Post-test paper and answers)

姓名：______________ 班級：______________ 日期：______________

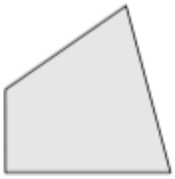
1. 在右圖中加畫**一條**直線，可分割成一個長方形和下列哪一種圖形？選擇適當的答案。



○ A. 等邊三角形 ○ B. 等腰三角形
○ C. 菱形 ○ D. 梯形

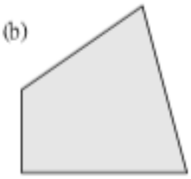
2. 在下面**各圖形**中**加畫一條虛線**，用**兩種不同的方法**把它們分割成**兩個圖形**。

(a)



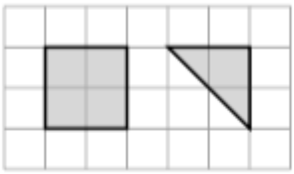
分割成：
一個 ______ 形
和一個 ________ 形

(b)



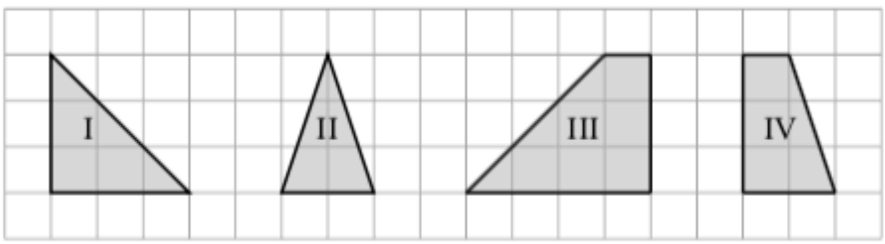
分割成：
兩個 ________ 形

3. 填空題。利用下圖中的一個三角形和一個正方形，可拼砌出哪一種**四邊形**？



答案：可拼砌出一個 ________ 形。

4. 下圖中，哪**兩個**圖形可拼砌出一個平行四邊形？選擇適當的答案。

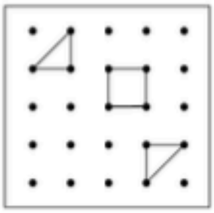


○ A. I 和 II ○ B. II 和 IV

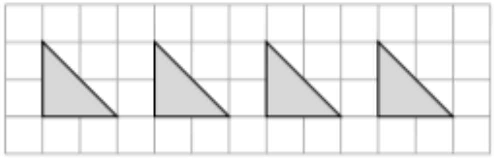
○ C. III 和 IV ○ D. I 和 III

5. 利用右圖中的兩個三角形和一個正方形，可以拼砌出**哪些四邊形**？

答案：可以拼砌出____。

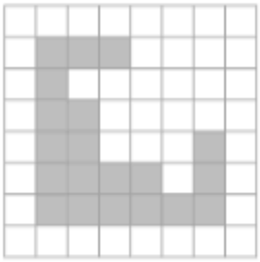


6. 利用右圖中的三角形，可拼砌出**哪些四邊形**？



答案：可拼砌出____。

7. 在右圖中加畫直線，把它分割成**四個形狀和大小都相同**的圖形。



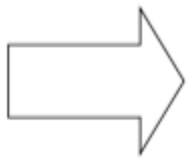
8. 下面有一張長方形方格紙（圖一），把它先分割成**兩個形狀和大小都相同**的圖形，再拼砌出（圖二）中空的正方形。在圖一及圖二中**加畫直線**，來表示你的分割及拼砌方法。

圖一 圖二

~ 全卷完 ~

姓名：____________ 班級：____________ 日期：____________

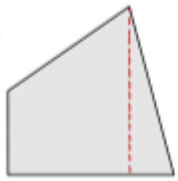
1. 在右圖中加畫**一條**直線，可分割成一個長方形和下列哪一種圖形？
選擇適當的答案。



○ A. 等邊三角形 ● B. 等腰三角形
○ C. 菱形 ○ D. 梯形

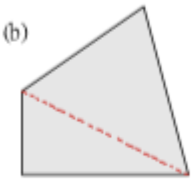
2. 在下面**各圖形**中**加畫一條虛線**，用**兩種不同的方法**把它們分割成**兩個圖形**。

(a)



分割成：
一個 梯 形
和一個 三角 形

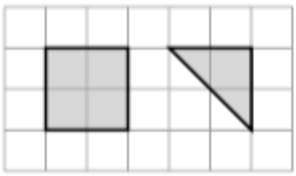
(b)



分割成：
兩個 三角 形

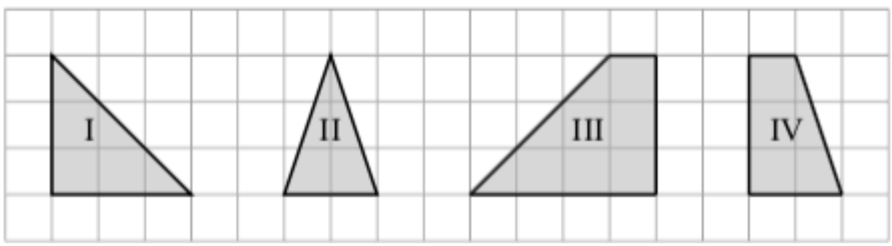
(答案合理即可)

3. 填空題。利用下圖中的一個三角形和一個正方形，可拼砌出哪一種**四邊形**？



答案：可拼砌出一個 梯 形。

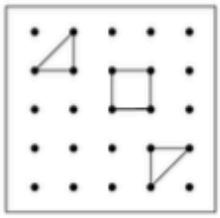
4. 下圖中，哪**兩個**圖形可拼砌出一個平行四邊形？選擇適當的答案。



○ A. I 和 II　　　　○ B. II 和 IV

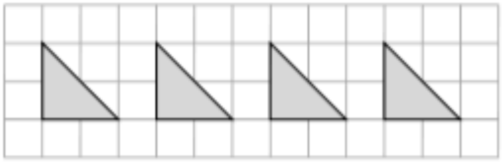
○ C. III 和 IV　　　● D. I 和 III

5. 利用右圖中的兩個三角形和一個正方形，可以拼砌出**哪些四邊形**？



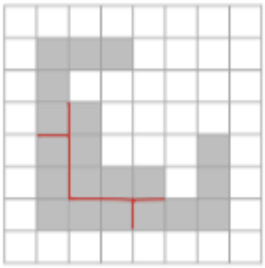
答案：可以拼砌出 長方形、平行四邊形、梯形 。

6. 利用右圖中的三角形，可拼砌出**哪些四邊形**？



答案：可拼砌出 梯形、長方形、平行四邊形、正方形 。

7. 在右圖中加畫直線，把它分割成**四個形狀和大小都相同**的圖形。



8. 下面有一張長方形方格紙（圖一），把它先分割成**兩個形狀和大小都相同**的圖形，再拼砌出（圖二）中空的正方形。在圖一及圖二中**加畫直線**，來表示你的分割及拼砌方法。

圖一　　　　圖二

~ 全卷完 ~