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Project Title

A Study of The Riemann Hypothesis From An Elementary Perspective

Contents

This is a project submitted to the Education University of Hong Kong for the degree of Bachelor of Education (Honours) (Secondary) in Mathematics on the seventh day of April of the year two thousand and twenty one.

DECLARATION

I, CHAN Ho Russell, declare that this report represents my own work under the supervision of Dr CHENG Kell Hiu Fai, and that it has not been submitted previously for examination to any tertiary institution.

CHAN HO RUSSELL

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ABSTRACT

This project aims to explain the famous Riemann Hypothesis from an elementary perspective so that undergraduate students with Mathematics background could understand more on this problem. We will first introduce some results in Complex Analysis so that the readers can understand the following chapters. Then, we examine how the functional equations of the Riemann zeta function were created. Next, we will explain the zeros of zeta function and the Riemann Hypothesis. Finally, we will also study the Generalized Riemann Hypothesis and some of its consequences.

1 Introduction

Ever since Riemann's paper "Üeber die Anzahl der Primzahlen unter einer gegebenen Grösse" (English translation: On the Number of Primes Less Than a Given Magnitude) was published in November 1859, countless Mathematicians have tried to solve the problem that comes with the paper, which is now known as the Riemann Hypothesis. This problem was then listed in Hilbert's 23 Mathematical Problems, which is a collection of mathematical problems whose solutions will take mankind to the furthering of that branch of Mathematics, in 1990 and the Millennium Prize Problems by the Clay Mathematics Institute in 2000, meaning that the first person who gives a correct proof or a counterexample to it will get one million US dollars from the Clay Mathematics Institute (Hilbert, 1902; Weisstein, n.d.; Clay Mathematics Institute, n.d.).

Many Mathematics students can state the Riemann Hypothesis, namely, all non-trivial zeros of the Riemann zeta function are on the critical line

$$\Re(s) = \frac{1}{2}.$$

Nevertheless, the development of the Riemann zeta function from Euler's definition in 1737,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

to its functional equation,

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{1-s}{2}}\zeta(1-s),$$

is hard for students to understand. This project aims to explain this problem from an elementary perspective so that undergraduate students with Mathematics background could also have a proper understanding of this famous problem. The first part of this paper is about some basic definitions and theorems in Complex Analysis. And then, we will discuss the functional equations of Riemann zeta function. After that, we will study the Riemann Hypothesis, the Generalized Riemann Hypothesis and some of its consequences.

2 Some Basic Definitions and Theorems

To study the Riemann Hypothesis, we should first study some results in Complex Analysis.

2.1 Basic Definitions In Complex Analysis

There are two representations of a complex number, namely the cartesian representation and the polar representation. For the cartesian representation, we write $z = x + iy$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. We denote $\Re(z)$ as the real part of z and $\Im(z)$ as the imaginary part of z. For the polar representation, we write $z = re^{i\theta}$, where $r \geq 0$ is the distance between z and the origin, and θ is the argument of z, which is the the angle inclined from the real axis. Also, we denote $\mathbb{C}$ as the set of all complex numbers and $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 2.1. *(Modulus of Complex Numbers)*
The modulus of $z = x + iy$ is defined as

$$|z| = \sqrt{x^2 + y^2}.$$

This is actually the distance between z and the origin. Further, notice that $z\bar{z} = |z|^2$, where $\bar{z}$ is the conjugate of z.

Definition 2.2. *(Complex-valued Functions)*
For $A \subset \mathbb{C}$, a mapping $f : A \to \mathbb{C}$ which assigns to each $z \in A$ a unique complex number $f(z)$ is called a complex-valued function.

Moreover, we can express $f(z)$ in terms of real-valued functions, that is $f(z) = u(z) + iv(z)$, where $u(z) = \Re(f(z))$ and $v(z) = \Im(f(z))$. Next, we shall discuss about the disks / discs and sets in $\mathbb{C}$.

Definition 2.3. *(Open Disks)*
An open disk centered at $a \in \mathbb{C}$ with radius $r > 0$ is defined as

$$D(a, r) = \{z \in C : |z - a| < r\}$$

Definition 2.4. *(Closed Disks)*
A closed disk centered at $a \in \mathbb{C}$ with radius $r > 0$ is defined as

$$\overline{D}(a, r) = \{z \in C : |z - a| \leq r\}.$$

Definition 2.5. *(Punctured Disks)*
A punctured disk centered at $a \in \mathbb{C}$ with radius $r > 0$ is defined as

$$D'(a, r) = \{z \in C : 0 < |z - a| < r\}.$$

An open disk is actually a collection of complex numbers in the disk except for its boundary $|z-a|=r$. A closed disk is an open disk with its boundary. Meanwhile, punctured disk is a collection of complex numbers in the open disk except for its center. Next, we shall take a look at the open sets and the closed sets.

Definition 2.6. *(Open Sets)*
The set $A \subset \mathbb{C}$ is open if $\forall z \in A$, $\exists\, r > 0$ such that $D(z,r) \subset A$.

Definition 2.7. *(Closed Sets)*
The set $A \subset \mathbb{C}$ is closed if $\mathbb{C} \backslash A$ is open.

Next, we shall also review the limit and continuity of complex-valued function. Similar to Real Analysis, their definitions are:

Definition 2.8. *(Limit of Complex-valued Functions)*
Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$ which is defined in A and $a \in \overline{A}$. Then

$$\lim_{z \to a} f(z) = w$$

if, $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that

$$z \in A \text{ and } 0 < |z-a| < \delta \;\Rightarrow\; |f(z) - w| < \epsilon.$$

Definition 2.9. *(Continuity of Complex-valued Functions)*
Let $A \subset \mathbb{C}$, $f : A \to \mathbb{C}$ which is defined in A and $a \in A$. Then f is continuous at $a \in A$ if $\forall \epsilon > 0$, $\exists\, \delta > 0$ such that

$$z \in A \text{ and } |z-a| < \delta \;\Rightarrow\; |f(z) - f(a)| < \epsilon.$$

2.2 Holomorphic Functions

In some complex analysis books, they remark that the study of complex analysis is a study of holomorphic functions. The concept of holomorphic functions is sort of like differentiability of functions in the complex world, but we allow the h to take complex values. Thus, instead of the right-hand limit and left-hand limit in real analysis, we have much more to consider. In this section, we shall study this important notion.

Definition 2.10. *(Differentiability at a Point)*
Let A be an open subset of $\mathbb{C}$ and $f : A \to \mathbb{C}$, then f is differentiable at $z \in A$ if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

Notice that $h \in \mathbb{C}$ and $z+h$ can approach z in any direction as $h \to 0$. Hence, we can not just consider the left-hand limit and right-hand limit.

Definition 2.11. *(Holomorphic Functions)*
A complex-valued function f is said to be a holomorphic function in an open set $A \subset \mathbb{C}$ if it is differentiable at every point on A. Moreover, we write $H(A)$ as the set of all holomorphic functions in A.

Notice that some textbooks use the term 'Analytic Function' instead of holomorphic function; they are interchangeable. Further to holomorphic functions, we also have entire functions. ***Entire Functions*** are functions that are holomorphic on the whole complex plane. Moreover, we have meromorphic functions as well.

Definition 2.12. *(Meromorphic Functions)*
A function f on an open set A is meromorphic function if is holomorphic on A except a set of poles (i.e. the points where f is not holomorphic at).

Next, we can look at some properties of holomorphic functions.

Theorem 2.13. *(Basic Properties)*
Let A be an open set, function f and g be holomorphic in A, and $a \in \mathbb{C}$. Then af, $f+g$, fg, and $f \circ g$ are also holomorphic in A. Moreover, $(f \circ g)'(z)$ also satisfies the Chain Rule, which means that

$$\frac{d}{dz} f(g(z)) = \frac{df}{dg}\frac{dg}{dz}.$$

By the above theorem, we know that if $f(z) = z$ is holomorphic, then any linear combination of z^n is holomorphic as well.

Theorem 2.14. *(Cauchy-Riemann Equations)*
Let A be an open subset of $\mathbb{C}$, $z = x + iy$, and $f : A \to \mathbb{C}$, $f(x,y) = u(x,y) + iv(x,y)$, where u and v are real-valued functions. If f is a holomorphic function on A, then u and v have first-order partial derivatives at every $z \in A$ and they satisfy the Cauchy-Riemann equations:

$$u_x = v_y \text{ and } u_y = -v_y.$$

Proof. By the definition of differentiability, we have

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. First, for $h \in \mathbb{R}$, we have

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y)-v(x,y)}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= u_x + iv_x \end{aligned}$$

Next, for $h \in i\mathbb{R}$, we let $h = ik$, $k \in \mathbb{R}$ and get

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{u(x,y+h)+iv(x,y+h)-u(x,y)-iv(x,y)}{ik} \\ &= \lim_{h\to 0} \frac{u(x,y+h)-u(x,y)}{ik} + i \lim_{h\to 0} \frac{v(x,y+h)-v(x,y)}{ik} \\ &= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ &= v_y - iu_y \end{aligned}$$

Comparing the results, we have $u_x = v_y$ and $v_x = u_y$. □

Notice that the contrapositive of the above theorem is "if f does not satisfy the Cauchy-Riemann equations then it is not holomorphic". This tools is useful for proving function that are non-holomorphic, but it is not sufficient to show otherwise.

2.3 Complex Series and Power Series

The Riemann Zeta Function is actually a complex series $\{n^{-s}\}$, where $s \in \mathbb{C}$. Therefore, it is essential to study some elementary properties of complex series.

To begin with, we shall state a few facts about the complex series. For complex series $\{a_n\} \in \mathbb{C}$, we have:

1. $\sum a_n$ is convergent $\iff \sum \Re(a_n)$ and $\sum \Im(a_n)$ are convergent.
2. If $\sum a_n$ is convergent, then $a_n \to 0$ as $n \to \infty$.
3. If $\sum a_n$ is convergent, then $\exists\ M \in \mathbb{R}$ such that $|a_n| < M$ for all n .
4. If $\sum a_n$ and $\sum b_n$ are convergent, then $\sum(sa_n + tb_n)$ is convergent for all $s, t \in \mathbb{C}$.
5. $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. If the series $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

To test whether a series converges, we can use the following tests:

I. Comparison Test

Let $\{a_n\}$ be a complex series, $\sum b_n$ be a converge series with $b_n \geq 0\ \forall n$, and $k > 0$. If $|a_n| \leq kb_n$ for all n, then $\sum a_n$ converges.

II. D'Alembert's Ratio Test

If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

exists and less than 1, then $\sum a_n$ converges. If the limit exists and it is greater than 1, then $\sum a_n$ diverges. If the limit exists and it is 1, this test gives no information.

III. Alternating Series Test

For series $\sum(-1)^n a_n$, where a_n is either positive $\forall n$ or negative $\forall n$, if a_n is monotonically decreasing and $a_n \to 0$ as $n \to \infty$, then $\sum(-1)^n a_n$ converges.

Next, we shall study some other definitions related to complex series.

Definition 2.15. *(Arithmetical Functions)*

A real- or complex-valued function defined on the set of positive integers is called an arithmetical function.

For example, the Euler's Totient Function $\phi(n)$, counting the number of positive integers less than n such that they are coprime to n, is an arithmetical function.

Definition 2.16. *(Dirichlet Series)*

A Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s},$$

where $s = \sigma + it$ and $f(n)$ is an arithmetical function.

Theorem 2.17. *(Absolute convergence of Dirichlet series)*

Suppose the series

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$$

does not converge for all s or diverge for all s. Then there exists a real number σ_a call the abscissa of absolute convergence, s.t. the series converges absolutely if $\sigma > \sigma_a$ but does not converge absolutely if $\sigma < \sigma_a$.

Proof. Let A be the set of all real σ so that

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right|$$

diverges. Then A is not empty because the series does not converge for all s. The set A is bounded above since the series does not diverge for all s. Thus, A has a least upper bound, say σ_a. If $\sigma < \sigma_a$, then $\sigma \in A$, otherwise σ would be an upper bound for A smaller than the least upper bound. If $\sigma > \sigma_a$, then $\sigma \notin A$ since σ_a is an upper bound for A. □

Definition 2.18. *(Power Series)*
A series of the form

$$\sum_{n=0}^{\infty} c_n(z-a)^n,$$

where $a, c_n \in \mathbb{C}$, is called a power series.

Definition 2.19. *(Radius of Convergence)*
The radius of convergence of a power series $\sum c_n(z-a)^n$ is defined as

$$R = \sup\{\ |z| : \sum |c_n(z-a)^n| \ \textit{converges}\ \}.$$

If $\sum |c_n(z-a)^n|$ converges for arbitrarily large $|z|$, we write $R = \infty$.

By this definition, we know that $\sum |c_n(z-a)^n|$ converges for all $|z| < R$ and $\sum |c_n(z-a)^n|$ will not converge for any $|z| > R$. Next, we should study the relationship between power series, radius of convergence and holomorphic function.

Theorem 2.20. *(Hadamard's Formula)*
If R is the radius of convergence of the power series $\sum c_n z^n$, then

$$\frac{1}{R} = \limsup |c_n|^{1/n}.$$

'lim sup' means if we cut the sequence $\{a_n\}$ at $n = N$ and consider only the sup of the rest of the sequence. For $N \to \infty$, the sup of the rest of the sequence is called lim sup, i.e.

$$\limsup a_n = \lim_{n\to\infty}\left(\sup_{N\geq n} a_N\right)$$

For example, even the sequence $\{(1+1/n)\sin(n)\}$ does not converge and is not stable when n is small, it oscillates between 1 and -1 when $n \to \infty$. In this case, we say $\limsup((1+1/n)\sin(n)) = 1$, and similarly, we have $\liminf((1+1/n)\sin(n)) = -1$.

Theorem 2.21. *The power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is holomorphic in in its disc of convergence (i.e. $D(0,R)$). Further, the derivative of f is the power series obtained by term-by-term differentiation:*

$$f'(z) = \sum_{n=0}^{\infty} n c_n z^{n-1}.$$

Proof. First, we should show that the series $g(z) = \sum_{n=0}^{\infty} nc_n z^{n-1}$ has the same radius of convergence as f. Note that $n^{1/n} \to 1$ as $n \to \infty$. Hence, we have $\limsup |c_n|^{1/n} = \limsup |nc_n|^{1/n}$, which implies f and g have the same radius of convergence.

Next, we shall show that $f' = g$. Suppose $|z| < r < R$ and let $f(z) = A_N(z) + B_n(z)$, where

$$A_N(z) = \sum_{n=0}^{N} c_n z^n \text{ and } B_N = \sum_{n=N+1}^{\infty} c_n z^n.$$

Then, we choose h such that $|z+h| < r$ and get

$$\frac{f(z+h) - f(z)}{h} - g(z) = \left(\frac{A_N(z+h) - A_N(z)}{h} - A_N'(z)\right) + (A_N'(z) - g(z)) + \left(\frac{B_N(z+h) - B_N(z)}{h}\right)$$

As $(a^n - b^n) = (a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$, we have

$$\left|\frac{B_N(z+h) - B_N(z)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z+h)^n - z^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |c_n|\, nr^{n-1}.$$

Since g converges absolutely on $|z| < r$, given $\epsilon > 0$, we can find $N_1 \in \mathbb{N}$ s.t. $N > N_1$ implies

$$\left|\frac{B_N(z+h) - B_N(z)}{h}\right| < \epsilon.$$

As

$$\lim_{N\to\infty} A_N'(z) = g(z),$$

we can find $N_2 \in \mathbb{N}$ s.t. $N > N_2$ implies $|A_N'(z) - g(z)| < \epsilon$.

As A_N' is just a derivative of polynomial, we can find $N > N_1, N_2$ and $\delta > 0$ s.t. $|h| < \delta$ implies

$$\left|\frac{A_N(z+h) - A_N(z)}{h} - A_N'(z)\right| < \epsilon$$

Thus, when $|h| < \delta$, we have

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| < 3\epsilon,$$

and the result follows. □

2.4 Integration on $\mathbb{C}$

Lastly, we should take a look at integration in the complex world. It is of interest that we will use tools in integration to prove the existence of high order derivative of holomorphic functions, or more precisely infinite differentiability (i.e. $f^{(n)}$ always exists regardless of the value of n). But before that, we should first study some important theorems in complex integration like Cauchy's Theorem. Similar to multi-variable calculus, we integrate functions along curves.

Definition 2.22. *(Curves)*

Let $[a,b] \subset \mathbb{R}$. *A function* $\gamma : [a,b] \to \mathbb{C}, \gamma(t) = x(t) + iy(t)$ *is a curve in* $\mathbb{C}$, *where* $t \in [a,b]$.

Furthermore, we say a curve is ***smooth*** if $\gamma'(z)$ exists, is continuous on $[a,b]$, and $\gamma'(t) \neq 0$ for $t \in [a,b]$. For the point $t = a$ and $t = b$, we define $\gamma'(a)$ and $\gamma'(b)$ as a one-sided limit. We say a curve is ***piecewise-smooth*** if γ is continuous on $[a,b]$ and there exists points $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ such that $\gamma(t)$ is smooth in $[t_j, t_{j+1}]$ for $j = 0, 1, 2, \ldots, n-1$.

Definition 2.23. *If* f *is a complex-valued function defined on a curve* γ, *then we define*

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j),$$

where $z_j = \gamma(t_j)$.

Notice that if a curve is smooth, then

$$\begin{aligned}\sum_{j=0}^{n-1} f(z_j)(z_{j+1} - z_j) &= \sum_{j=0}^{n-1} f(\gamma(t_j)) \frac{\gamma(t_{j+1}) - \gamma(t_j)}{t_{j+1} - t_j}(t_{j+1} - t_j) \\ &\to \int_a^b f(\gamma(t))\gamma'(t)dt \quad \text{as } n \to \infty\end{aligned}$$

Definition 2.24. *(Primitive For a Function)*

Suppose f *is a function on an open set* A. *The function* F *which is holomorphic on* A *such that* $F'(z) = f(z)$ *is called a primitive for* f *on* A.

Theorem 2.25. *If a continuous function* f *has a primitive* F *on an open set* A, *and* γ *is a curve in* A *such that* $\gamma : [a,b] \to \mathbb{C}$, *then*

$$\int_\gamma f(z)dz = F(b) - F(a).$$

Proof. If γ is smooth, then

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_a^b F'(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \frac{d}{dt}F(\gamma(t))dt \\ &= F(b) - F(a)\end{aligned}$$

The third line is an application of the Chain Rule. If γ is piecewise-smooth, then similarly we have

$$\begin{aligned}\int_\gamma f(z)dz &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\gamma'(t)dt \\ &= \sum_{k=0}^{n-1} (F(\gamma(t_{k+1})) - F(\gamma(t_k))) \\ &= F(\gamma(t_n)) - F(\gamma(t_0)) \\ &= F(b) - F(a)\end{aligned}$$

□

Furthermore, a holomorphic function in an open disk has a primitive in that disk. Now, with the above results, we can now study Cauchy's work.

Theorem 2.26. *(Cauchy's theorem on a Closed Curve on a Disk)*
If f is holomorphic in a disk, then for any closed curve γ in that disk, we have

$$\int_\gamma f(z)dz = 0.$$

Proof. Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve in that disk. Then we have $\gamma(a) = \gamma(b)$. As f is holomorphic in an open disk, it has a primitive in that disk. Thus,

$$\int_\gamma f(z)dz = F(\gamma(a)) - F(\gamma(b)) = 0.$$

□

Cauchy extended his theory by consider the toy contour of closed curve and proved the following theorem, which is now known as Cauchy's Theorem.

Theorem 2.27. *(Cauchy's Theorem)*
Let A be a simply connected open set (i.e. no holes) in $\mathbb{C}$, f be holomorphic in A, and $\gamma \subset A$ be a smooth closed curve. Then,

$$\int_\gamma f(z)dz = 0.$$

With this theorem, Cauchy discovered what is now known as the Cauchy Integral Formula.

Theorem 2.28. *(Cauchy Integral Formula)*
Suppose f is holomorphic in a simply connected open set D. If D is bounded by γ, then for all $w \in D$

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz.$$

He proved the above formula by considering a disk with center w and radius ϵ. The proof is very complicated and is left as an extended reading for the readers. (The proof can be found in most of the Complex Analysis textbooks, e.g. Stein and Shakarchi's (2002) Complex Analysis page 45 to 47).

Finally, we can show the following powerful property, that is, the existence of high order derivative of holomorphic function.

Theorem 2.29. *(Infinite Differentiability of Holomorphic Function)*
If f is holomorphic on an open set A, then there exists $f^{(n)}$ for all $n \in \mathbb{N}$ on A.

To prove this, we first show that the existence of f' implies the existence of f'' , and then by induction we have that $f^{(n)}$ exists.

Proof. Let $a \in A$, $\gamma \subset A$ and choose $r > 0$ so that $\bar{D}(a,r) \subset A$. Since f is holomorphic, f' exists. For $|h| < r$, we have

$$\begin{aligned}
\frac{f'(a+h) - f'(a)}{h} &= \frac{1}{2\pi i h} \int_\gamma \frac{f(z)}{(z-a-h)^2} dz - \frac{1}{2\pi i h} \int_\gamma \frac{f(z)}{(z-a)^2} dz \\
&= \frac{1}{2\pi i h} \int_\gamma f(z) \left(\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2} \right) dz \\
&= \frac{1}{2\pi i} \int_\gamma f(z) \left(\frac{2(z-a) - h}{(z-a-h)^2 (z-a)^2} \right) dz \\
&\to \frac{1}{2\pi i} \int_\gamma f(z) \left(\frac{2(z-a)}{(z-a)^2 (z-a)^2} \right) dz \quad \text{as } h \to 0 \\
&= \frac{1}{2\pi i} \int_\gamma \frac{2f(z)}{(z-a)^3} dz
\end{aligned}$$

Hence, f' exists implies that f'' exists. By induction, we know that $\exists\ f^{(n)} \forall n \in \mathbb{N}$. □

One may notice that we did not cancel the 2 in the last equation, it is because we would like to further derive the formula for the n-th derivatives, that is

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

After studying some important definitions and theorems in complex analysis, we can now move on to the functional equations of the Riemann zeta functions.

3 The Functional Equations of Riemann Zeta Function

Before Riemann's paper, Euler discovered some properties of the series $\sum n^{-x}$ for $x > 1$. Based on this discovery, Riemann extended the input of this function from real variable to complex variable and denoted $\zeta(s) = \sum n^{-s}$ for $\Re(s) > 1$. The next reasonable question to ask is that whether we can further extend the domain of ζ to the whole complex plane. Riemann answered this question with a technique called analytic continuation and the key of this technique is functional equation. In this chapter, we will study the development of the functional equations for the Riemann zeta function and see how he extended the domain of ζ.

3.1 Euler's Observation I

In Euler's (1737) paper "Variae observationes circa series infinitas" (English translation: Several Remarks on Infinite Series), he showed that the infinite series

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

can be express in terms of an infinite product of prime numbers

$$\left(\frac{2^s}{2^s-1}\right)\left(\frac{3^s}{3^s-1}\right)\left(\frac{5^s}{5^s-1}\right)\left(\frac{7^s}{7^s-1}\right)\left(\frac{11^s}{11^s-1}\right)\cdots = \prod_{p \text{ prime}} \frac{p^s}{p^s-1}$$

Theorem 3.1. *(Euler's Observation)*
For $s \in \mathbb{R}$ and $s > 1$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{p^s}{p^s-1}.$$

Proof. Let

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \tag{1}$$

(1) $\times \frac{1}{2^s}$, we have

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots \tag{2}$$

(1) − (2), we get

$$\frac{2^s-1}{2^s}\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots \tag{3}$$

(3) $\times \frac{1}{3^s}$ we have

$$\frac{1}{3^s} \times \frac{2^s-1}{2^s}\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \cdots \tag{4}$$

(3) − (4), we get

$$\frac{3^s-1}{3^s} \times \frac{2^s-1}{2^s}\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots.$$

Repeat the procedure with each prime number, we will have

$$\prod_{p \text{ prime}} \left(\frac{p^s - 1}{p^s}\right) \times \zeta(s) = 1.$$

Making $\zeta(s)$ as the subject, we have

$$\zeta(s) = \prod_{p \text{ prime}} \left(\frac{p^s}{p^s - 1}\right) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

□

Actually, Euler (1737) did not use the notation ζ in his paper, he just wrote the whole series for many times. A hundred years later, Riemann (1859) used ζ to represent this series and made some new breakthroughs. That is why this series is called 'Riemann zeta function'. In Riemann's paper, he first changed the domain of ζ from real number greater than 1 to complex number with $\Re(s) > 1$. By Euler's observation, we have the following theorem:

Theorem 3.2. *For $s \in \mathbb{C}$ and $\Re(s) > 1$, we have*

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Furthermore, to extend the domain of ζ to the whole complex plane, we need analytic continuation. Analytic continuation is an extension of domain of a function from the original set to a bigger set. In general, we have the following definition:

Definition 3.3. *(Analytic Continuation)*
Let functions f and g be analytic functions on the domain A and B respectively with $A \cap B$ being non-empty. If $f = g$ on $A \cap B$, then we say that g is an analytic continuation of f to B or f is an analytic continuation of g to A.

In our case, Riemann aimed to extend the domain of ζ from the set $\{s \in \mathbb{C} : \Re(s) > 0\}$ to a bigger set $\mathbb{C}$, which is the set of all complex number.

3.2 Functional Equation of Riemann Zeta Function I

Before we look at the first functional equation of Riemann zeta function and its proof, we shall first study the Jacobi's ϑ function, the ψ function and the Gamma function. The ϑ and ψ function will be used in proving the functional equation while the Gamma function is actually a part of the equations.

Definition 3.4. *(Jacobi's ϑ function)*
For $s \in \mathbb{C}$ and $\Re(s) > 0$, we define

$$\vartheta(s) = \sum_{n \in \mathbb{Z}} e^{-\pi s n^2}.$$

Furthermore, the ϑ function has the functional equation $\vartheta(x) = \frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right)$.

Lemma 3.5. *For $s \in \mathbb{C}$ and $\Re(s) > 0$, if we define*

$$\psi(s) = \sum_{n\in\mathbb{N}} e^{-\pi s n^2},$$

then

$$\vartheta(s) = 2\psi(s) + 1.$$

Proof.

$$\begin{aligned}
\vartheta(s) &= \sum_{n\in\mathbb{Z}} e^{-\pi s n^2} \\
&= \sum_{n=-\infty}^{-1} e^{-\pi s n^2} + e^{-\pi s (0)^2} + \sum_{n=1}^{+\infty} e^{-\pi s n^2} \\
&= 2\sum_{n=1}^{+\infty} e^{-\pi s n^2} + 1 \\
&= 2\sum_{n\in\mathbb{N}} e^{-\pi s n^2} + 1 \\
&= 2\psi(s) + 1
\end{aligned}$$

□

Lemma 3.6. *For $s \in \mathbb{C}$ and $\Re(s) > 0$, we have*

$$\psi(s) = \frac{1}{\sqrt{s}}\psi\left(\frac{1}{s}\right) - \frac{1}{2} + \frac{1}{2\sqrt{s}}.$$

Proof. By the functional equation of ϑ function and Lemma 3.5, we have

$$\begin{aligned}
\vartheta(s) &= \frac{1}{\sqrt{s}}\vartheta\left(\frac{1}{s}\right) \\
2\psi(s) + 1 &= \frac{1}{\sqrt{s}}\left(2\psi\left(\frac{1}{s}\right) + 1\right) \\
\psi(s) + \frac{1}{2} &= \frac{1}{\sqrt{s}}\psi\left(\frac{1}{s}\right) + \frac{1}{2\sqrt{s}} \\
\psi(s) &= \frac{1}{\sqrt{s}}\psi\left(\frac{1}{s}\right) - \frac{1}{2} + \frac{1}{2\sqrt{s}}.
\end{aligned}$$

□

Next, we shall study the Gamma function which is an important part of the functional equations of the zeta function.

Definition 3.7. *(Gamma Function)*

For $s \in \mathbb{C}$ and $\Re(s) > 0$, we define

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.$$

Same as the zeta function, we would like to extend the domain of Γ. By integration by part, we have

$$\begin{aligned}\Gamma(s) &= \int_0^\infty t^{s-1}e^{-t}dt \\ &= -(t^{s-1}e^{-t})\Big|_0^\infty + \int_0^\infty (s-1)t^{s-2}e^{-t}dt \\ &= (s-1)\Gamma(s-1)\end{aligned}$$

By the above functional equation, we can extend the domain of Gamma from $\{s \in \mathbb{C} : \Re(s) > 0\}$ to $\{s \in \mathbb{C} : \Re(s) > -1 \text{ and } s \neq 0\}$. Applying the same manner, we can extend the domain of Gamma to $\mathbb{C}\backslash\{0, -1, -2, -3, \ldots\}$. Furthermore, by definition, we have $\Gamma(1) = 1$. Hence, one can easily see that $\Gamma(n) = (n-1)!\ \forall n \in \mathbb{N}$. Although the Gamma function is not an entire function, the reciprocal of it is differentiable everywhere. Before we show that it true, we shall first study the following two lemmas.

Lemma 3.8. *For* $0 < \Re(s) < 1$,

$$\int_0^\infty \frac{\lambda^{s-1}}{1+\lambda}d\lambda = \frac{\pi}{\sin(\pi s)}.$$

Since the proof involves some of the advanced topic in Complex Analysis like contour integration and residue formula, we skip it here. Moreover, it suffices to show the equality holds for $0 < \Re(s) < 1$ as it will then hold on $\mathbb{C}$ by analytic continuation. The next lemma states the following:

Lemma 3.9. *For all* $s \in \mathbb{C}$,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Proof. For $0 < \Re(s) < 1$,

$$\Gamma(1-s) = \int_0^\infty e^{-u}u^{-s}du.$$

For $t > 0$, let $u = \lambda t$ and get

$$\Gamma(1-s) = t\int_0^\infty e^{-\lambda t}(\lambda t)^{-s}d\lambda.$$

Next, consider $\Gamma(s)\Gamma(1-s)$,

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t}t^{s-1}dt\ \Gamma(1-s) \\ &= \int_0^\infty e^{-t}t^{s-1}\Gamma(1-s)dt \\ &= \int_0^\infty e^{-t}t^{s-1}t\int_0^\infty e^{-\lambda t}(\lambda t)^{-s}d\lambda\ dt \\ &= \int_0^\infty\int_0^\infty e^{-\lambda t-t}\lambda^{-s}d\lambda\ dt \\ &= \int_0^\infty\int_0^\infty e^{-\lambda t-t}\lambda^{-s}dt\ d\lambda.\end{aligned}$$

Notice that

$$\begin{aligned}\int_0^\infty e^{-\lambda t-t}\lambda^{-s}dt &= \frac{\lambda^{-s}}{1+\lambda}\int_0^\infty e^{-t(1+\lambda)}d(t(1+\lambda)) \\ &= \frac{\lambda^{-s}}{1+\lambda}\left(-e^{-t(1+\lambda)}\right)\Big|_0^\infty \\ &= \frac{\lambda^{-s}}{1+\lambda}.\end{aligned}$$

Hence,

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_0^\infty \frac{\lambda^{-s}}{1+\lambda} d\lambda \\ &= \int_0^\infty \frac{\lambda^{(1-s)-1}}{1+\lambda} d\lambda \\ &= \frac{\pi}{\sin(\pi - s\pi)} \\ &= \frac{\pi}{\sin(\pi s)}.\end{aligned}$$

By analytic continuation of Γ, the result follows. □

Changing the subject of the above equation, we have

$$\frac{1}{\Gamma(s)} = \frac{\sin(\pi s)\Gamma(1-s)}{\pi}.$$

Note that the poles of $\Gamma(s)$ are cancelled by the sine function. Hence, $1/\Gamma$ can be computed by the above formula for non-positive integers. Thus, $1/\Gamma$ is entire with zeros at $s = 0, -1, -2, \ldots$. The last lemma before we introduce the functional equation states the following:

Lemma 3.10. *For* $s \in \mathbb{C}\backslash\{0, 1\}$,

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx = \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx + \frac{1}{s(s-1)}.$$

Proof. By the lemma 3.6, we have

$$\begin{aligned}\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx &= \int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right)dx \\ &= \int_0^1 \left(x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right) + \frac{1}{2}\left(x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1}\right)\right)dx \\ &= \int_0^1 x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{2}\left[\frac{1}{\frac{s}{2}-\frac{1}{2}}x^{\frac{s}{2}-\frac{1}{2}} - \frac{1}{\frac{s}{2}}x^{\frac{s}{2}}\right]_0^1 \\ &= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{s(s-1)}.\end{aligned}$$

Next, we substitute $x = \frac{1}{y}$. Note that $dx = -y^{-2}dy$ and as $x = 0$, $y \to \infty$; $x = 1$, $y = 1$. Hence, we have

$$\begin{aligned}\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx &= \int_\infty^1 \left(\frac{1}{y}\right)^{\frac{s}{2}-\frac{3}{2}}\psi(y)\left(-\frac{1}{y^2}\right)dy + \frac{1}{s(s-1)} \\ &= \int_1^\infty \left(\frac{1}{x}\right)^{\frac{s}{2}-\frac{3}{2}}\psi(x)\frac{1}{x^2}dx + \frac{1}{s(s-1)} \\ &= \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx + \frac{1}{s(s-1)}.\end{aligned}$$

□

Now that we have all the tools needed, we can study the following important theorem by Riemann (1859). Although he completed this proof in 5 lines, we shall provide a more detailed proof so that normal undergrad students can understand.

Theorem 3.11. *(Functional Equation of Riemann Zeta Function)*
For $s \in \mathbb{C}\backslash\{0,1\}$, we have

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Proof. We shall begin our proof with the Gamma function:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$$
$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{s/2-1}e^{-t}dt.$$

Next, we let $t = \pi n^2 x$. Note that $dt = \pi n^2 dx$ and when $t = 0$, $x = 0$; when $t \to \infty$, $x \to \infty$ as well. Hence, we have

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty \left(\pi n^2 x\right)^{s/2-1}e^{-\pi n^2 x}\pi n^2 dx \\ &= \int_0^\infty \pi^{s/2-1}n^{s-2}x^{s/2-1}e^{-\pi n^2 x}\pi n^2 dx \\ &= \int_0^\infty \pi^{s/2}n^s x^{s/2-1}e^{-\pi n^2 x}dx \\ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} &= \int_0^\infty x^{s/2-1}e^{-\pi n^2 x}dx.\end{aligned}$$

Then, for $\Re(s) > 1$, we define

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Next, summation over n gives

$$\begin{aligned}\sum_{n=1}^\infty \left[\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s}\right] &= \sum_{n=1}^\infty \int_0^\infty x^{s/2-1}e^{-\pi n^2 x}dx \\ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^\infty \frac{1}{n^s} &= \int_0^\infty x^{s/2-1}\sum_{n=1}^\infty e^{-\pi n^2 x}dx \\ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{s/2-1}\psi(x)dx \\ \xi(s) &= s(s-1)\int_0^\infty x^{s/2-1}\psi(x)dx.\end{aligned}$$

Next, breaking the integrals into two parts yields

$$\begin{aligned}\xi(s) &= s(s-1)\int_1^\infty x^{s/2-1}\psi(x)dx + s(s-1)\int_0^1 x^{s/2-1}\psi(x)dx \\ &= s(s-1)\int_1^\infty x^{s/2-1}\psi(x)dx + s(s-1)\left(\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx + \frac{1}{s(s-1)}\right) \quad \text{(Lemma 3.10)} \\ &= s(s-1)\int_1^\infty \left(x^{s/2-1} + x^{-s/2-1/2}\right)\psi(x)dx + 1 \\ &= s(s-1)\int_1^\infty \frac{1}{x}\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)\psi(x)dx + 1.\end{aligned}$$

As ψ has exponential decay at ∞, i.e. the $\limsup(\psi(s))$ is bounded, ξ is an entire function. Finally, notice that the R.H.S. of the above equation remain the same when the input of ξ is $1-s$ instead of s:

$$\begin{aligned}\xi(1-s) &= (1-s)(1-s-1)\int_1^\infty \frac{1}{x}\left(x^{\frac{1-s}{2}} + x^{\frac{1-1+s}{2}}\right)\psi(x)dx + 1 \\ &= s(s-1)\int_1^\infty \frac{1}{x}\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)\psi(x)dx + 1 \\ &= \xi(s).\end{aligned}$$

Thus, we have

$$\begin{aligned}\xi(s) &= \xi(1-s) \\ s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= (1-s)(1-s-1)\pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).\end{aligned}$$

Moreover, as $\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, we have

$$\zeta(s) = \frac{\xi(s)\pi^{s/2}}{s(s-1)\Gamma(s/2)}.$$

As ξ and $1/\Gamma$ is entire function, the only poles for ζ are the point 0 and 1. □

Recall that $1/\Gamma(s) = 0$ for $s = 0, -1, -2, \dots$. It immediately follows that $\zeta(s) = 0$ for $s = -2, -4, \dots$, which is call the trivial zeros of zeta function. We will discuss the zeros of zeta function again in chapter 4. After Riemann had proven the above functional equation, he showed another functional equation which makes the trivial zeros more apparent to us. To show the second functional equation, the point of departure is another observation by Euler.

3.3 Euler's Observation II

31 years after "Variae observationes circa series infinitas", Euler (1768) published another paper which is very useful for our topic. The title of this paper is "Remarques sur un beau rapport entre les series des puissances tant directes que reciproques" (English translation: Remarks on a beautiful relation between direct as well as reciprocal power series).

In his paper, Euler (1768) stated the following:

Theorem 3.12. *For $s \in \mathbb{Z}$,*

$$\frac{1 - 2^{s-1} + 3^{s-1} - 4^{s-1} + 5^{s-1} - 6^{s-1} + \cdots}{1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots} = \frac{-1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (s-1)(2^s - 1)}{(2^{s-1} - 1)\pi^s}\cos\left(\frac{\pi s}{2}\right).$$

To simplify the above equation, we define

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

Then, with the eta and Gamma functions, we get:

$$\frac{1-2^{s-1}+3^{s-1}-4^{s-1}+5^{s-1}-6^{s-1}+\cdots}{1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+\cdots} = \frac{-1\cdot 2\cdot 3\cdot 4\cdot\cdots\cdot(s-1)(2^s-1)}{(2^{s-1}-1)\pi^s}\cos\Big(\frac{\pi s}{2}\Big)$$
$$\frac{\eta(1-s)}{\eta(s)} = -\frac{\Gamma(s)(2^s-1)}{(2^{s-1}-1)\pi^s}\cos\Big(\frac{\pi s}{2}\Big).$$

3.4 Functional Equation of Riemann Zeta Function II

To show the second functional equation,

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\Big(\frac{s\pi}{2}\Big),$$

is true, one shall first notice that

$$\begin{aligned}(1-2^{1-s})\zeta(s) &= \zeta(s)-2^{1-s}\zeta(s)\\ &= \sum_{n=1}^{\infty} n^{-s} - 2\sum_{n=1}^{\infty}(2s)^{-s}\\ &= 1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+\cdots\\ (1-2^{1-s})\zeta(s) &= \eta(s). \qquad (5)\end{aligned}$$

Then, by Theorem 3.12, we have

$$\begin{aligned}\frac{\eta(1-s)}{\eta(s)} &= -\frac{\Gamma(s)(2^s-1)}{(2^{s-1}-1)\pi^s}\cos\Big(\frac{\pi s}{2}\Big)\\ \frac{(1-2^s)\zeta(1-s)}{(1-2^{1-s})\zeta(s)} &= -\frac{\Gamma(s)(2^s-1)}{(2^{s-1}-1)\pi^s}\cos\Big(\frac{\pi s}{2}\Big)\\ \zeta(1-s) &= \frac{2}{(2\pi)^s}\Gamma(s)\zeta(s)\cos\Big(\frac{\pi s}{2}\Big).\end{aligned}$$

Now, if we put s instead of $1-s$ to the zeta function on the R.H.S., we will have

$$\begin{aligned}\zeta(s) &= \frac{2}{(2\pi)^{1-s}}\Gamma(1-s)\zeta(1-s)\cos\Big(\frac{\pi(1-s)}{2}\Big)\\ \zeta(s) &= 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\Big(\frac{s\pi}{2}\Big).\end{aligned}$$

However, as $\zeta(1)$ is undefined, the input cannot be 0 and 1. Also, the value of the $\zeta(s)$ is depending on $\zeta(1-s)$ and we only know the values of $\zeta(s)$ for $\Re(s)>1$ from the original definition. Therefore, this functional equation may not be useful in $0\le\Re(s)\le 1$. Also, as the domain of Γ is the set of all complex number except non-positive integers, the domain of $\Gamma(1-s)$ is $\mathbb{C}\backslash\mathbb{N}$. Therefore, the above functional equation is useful in $\mathbb{C}\backslash\{0,1,2,\dots\}\cup\{s\in\mathbb{C}: 0<\Re(s)<1\}$. To make it more elegant, many textbooks simply state the following:

For $\Re(s)<0$, the functional equation for ζ is

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\Big(\frac{s\pi}{2}\Big).$$

On the other hand, from equation (5), we can see that

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}.$$

By the Alternating Series Test (section 2.3), one can easily show that $\eta(s)$ is convergent for $\Re(s) > 0$. Hence, the above equation is also an extension of zeta function from $\Re(s) > 1$ to $\Re(s) > 0$ with a pole at the point 1 due to the denominator.

3.5 Short Summary

In short, there are four equations for the zeta functions:

For $\Re(s) > 1$, we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{p^s}{p^s - 1}. \tag{6}$$

For $\mathbb{C}\backslash\{0, 1\}$, we have

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \tag{7}$$

For $\Re(s) < 0$, we have

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \zeta(1-s) \sin\left(\frac{s\pi}{2}\right). \tag{8}$$

For $\Re(s) > 0$ and $s \neq 1$, we have

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}. \tag{9}$$

4 Zeros of The Zeta Function: The Riemann Hypothesis

When we talk about the zeros of the zeta function, we mean the inputs of the function that make $\zeta = 0$. From Equation (8) of the last chapter, it is clear that for negative even integers, $\zeta(s) = 0$ as $\sin s = 0$ for $s = k\pi$, for $k \in \mathbb{Z}$. This is also known as the trivial zero of the zeta function. Apart from the trivial zeros, are there non-trivial zero of the zeta function?

For $\Re(s) > 1$, we can see from Equation (6) that the values of ζ are products of positive numbers. Hence, there are no zeros in this region.

For $\Re(s) < 0$, we can take a look at Equation (8). $2^s\pi^{s-1}$ is obviously positive for all s in this region. The Gamma function is also non-zero for all s in this region. $\zeta(1-s)$ in this region is actually same as $\zeta(s)$ in $\Re(s) > 1$, implying non-zero in $\Re(s) < 0$ as well. Also, zeros of the sine function create the trivial zeros only. Hence, we can conclude that there are no non-trivial zeros in $\Re(s) < 0$.

The remaining region is $0 \leq \Re(s) \leq 1$, which is called the ***Critical Strip***. Riemann suggested that may be all the non-trivial zeros are on the line $\Re(s) = 1/2$, which is called the ***Critical Line***. This conjecture is now known as ***The Riemann Hypothesis***. Riemann did not prove this statement, he explained: "ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläiufig bei Seite gelassen, da er für den nächsten Zweek meiner Untersuchung entbehrlich schien" (English translation: "I have meanwhile temporarily put aside the search for [the proof] after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation") (Riemann, 1859, p. 139). Since then, countless of Mathematicians have attempted to prove it, but none of them has succeeded.

5 Generalized Riemann Hypothesis and Its Consequences

In this final chapter, we will talk about the Generalized Riemann Hypothesis and some of its consequences, namely the weak form of the famous Goldbach's Conjecture. But before we introduce this new conjecture, we should first take a look on the Dirichlet charter χ mod k and the Dirichlet L-series $L(s, \chi)$.

Definition 5.1. *(Dirichlet Characters)*
A Dirichlet characters (mod k) is an arithmetical function $\chi : \mathbb{N} \to \mathbb{C}$ *satisfying:*

1. $\chi(mn) = \chi(m)\chi(n), \ \forall m, n \in \mathbb{N}$
2. $|\chi(n)| = \begin{cases} 1 & \textit{if } (n,k) = 1 \\ 0 & \textit{otherwise} \end{cases}$
3. $\chi(n + km) = \chi(n), \ \forall n, m \in \mathbb{N}$
4. $\chi^{\phi(k)}(n) = 1, \ (n,k) = 1,$ *where* ϕ *is the Euler's totient function*

This character is like a filter for relative prime. Non-relative prime input will becomes zero while relative prime input will be preserved. Next, for the Dirichlet L-series, we have the following definition:

Definition 5.2. *(The Dirichlet L-series)*
For $\Re(s) > 1$*, the Dirichlet L-series is defined as*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Notice that Riemann zeta function is also a Dirichlet L-series with $k = 1$ and $\chi(n) = 1 \ \forall n \in \mathbb{N}$.

Conjecture 5.3. *(Generalized Riemann Hypothesis)*
For all Dirichlet characters, all the non-trivial zero of Dirichlet L-series lies on the critical line.

As $\zeta(s)$ is a Dirichlet L-series, the Generalized Riemann Hypothesis implies Riemann Hypothesis.

On the other hand, the Generalized Riemann Hypothesis also implies the weak form of the Goldbach's Conjecture (some textbooks refer it as odd Goldbach's Conjecture or the 3-primes problem), which is also an important hypothesis in Number Theory. In 1742, Christian Goldbach suggested the following:

Conjecture 5.4. *(The Goldbach's Conjecture (Weak Form))*
For all odd number greater than 7, it can be expressed as the sum of three odd prime numbers.

Luckily, this conjecture, unlike the Riemann Hypothesis, has been proved in 2013 by Prof. Harald Andrés Helfgott.

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