The Education University of Hong Kong

Department of Mathematics and Information Technology

Bachelor of Education (Honours) (Secondary) in Mathematics

Honours Project

Project title: Comparing the UK advanced level contents and M2 contents in linear algebra

Project supervisor: Dr. Yuen Man Wai

Submitted by: Chiu Hing Lun

Date of submission: 8 April 2021

Declaration

I, Chiu Hing Lun declare that this research report represents my own work under the supervision of Assistant professor Dr. Yuen Man Wai, and that it has not been submitted previously for examination to any tertiary institution.

Chiu Hing Lun

8th April, 2021

Abstract

This research project compares the syllabus and textbook content in linear algebra of HKDSE M2 with mathematics qualifications in the UK. The syllabus and textbooks in the UK cover much more topics in vectors and matrices than those in Hong Kong. The differences in topics, representation and textbook characteristics are organized in tables and presented in this research. Moreover, some opinions of both the undergraduates who studied HKDSE M2 and those who have taken CIE or Edexcel IALs mathematics papers are organized in charts. Some recommendations for HKDSE M2 textbooks, syllabus and curriculum are suggested after that for enhancing the teaching contents in linear algebra, for example implementing matrix transformation in the syllabus, replacing some proofs from the textbooks to the supplementary notes and considering the extended part of mathematics as regular electives. However, inadequate research support and small sample size may hinder the development of this research. For further investigation, comparing local curriculum with those in other countries, such as Singapore, the United State or Canada in different topics of mathematics to lead Hong Kong mathematics curriculum into introspection and substantial improvement. More importantly, the awareness of linear algebra is hoped to elevate in local mathematics education and more learning hours and investigations should be implemented in HKDSE M2.

Table of content

Introduction

Linear algebra is one of the most important topics in mathematics. It can be applied in many aspects and the importance keeps increasing nowadays. Matrix algebra demonstrates how powerful algebraic notation can be by generalizing the single-variable algebra. The vector spaces and linear transformation lead linear algebra also connect with geometry, algebra, statistics and analysis (Tucker, 1993). It is covered in the subject of Mathematics Module 2 Algebra and Calculus (HKDSE M2), which is the secondary advanced mathematics education in Hong Kong. If the local students want to study in foreign countries, such as the United Kingdom, Singapore or the United State and pursue professional training opportunities or admission to some international universities, they may choose to take some board-scaled examinations so that the results can be globally recognized. Both Cambridge International Examination (CIE) and Edexcel International Advanced Levels (Edexcel IALs) are examples of globally recognized examinations developed in the UK. More importantly, the local high-school mathematics education can make some progress or reform through comparing with the curriculum of other countries so that the academic level of Hong Kong students can be enhanced to be globally recognized. The documents provided by ITS Education in Hong

Kong state that the similarity of the contents between Further Pure Mathematics Papers of Edexcel IALs and the HKDSE M2 is approximately 20-35%. With the comparison of mathematics teaching contents between international qualifications and that of HKDSE, more improvement can be implemented in Hong Kong mathematics education to enhance the overall academic level in mathematics. Moreover, Hong Kong Certificate of Education Examination, which was equivalent to the General Certificate of Secondary Education in the UK, was replaced by HKDSE in 2012. The education reform in Hong Kong is one of the reasons to choose UK math content to compare with Hong Kong math content.

This research in mathematics only involves mathematics content in different qualifications and does not cover any other subjects. This essay is going to compare the contents in linear algebra of HKDSE M2 with the Pure Mathematics papers (P) and Further Pure Mathematics papers (FP) of UK qualifications, which refers to CIE and Edexcel IALs altogether. The syllabus and textbook content which cover the linear algebra contents of different qualifications are compared into some aspects. Furthermore, some opinions of both the undergraduates who studied HKDSE M2 and those who have taken CIE or Edexcel IALs mathematics papers will be collected. After that, some recommendations will be suggested for enhancing the teaching contents in linear algebra of the HKDSE M2 textbooks or syllabus.

Research Questions/ Purpose

In order to recommend some progress in Hong Kong mathematics education through comparison, interview and questionnaire, some issues are discussed in the following. The main research questions of this study are:

1. What are the differences between the syllabus and textbook content which cover the topic of linear algebra of CIE, Edexcel IALs and HKDSE M2?
2. What is the view of undergraduates who took UK qualifications and HKDSE M2 by knowing the difference between them? What is the level of their knowledge of linear algebra?
3. What suggestions can be made by the local students to mathematics education in Hong Kong, syllabus or textbooks of HKDSE M2?
4. From the difference of the contents and the opinions collected, how to improve or advance the contents of textbooks in Hong Kong?
5. How can teachers help local students to explore linear algebra?

This study aims to improve the understanding of students in linear algebra by developing the difference between the mathematics of UK qualifications and HKDSE M2. Moreover, it is beneficial for the HKDSE M2 textbooks to advance the contents of linear algebra for students to explore, but not just for memorizing the concepts or calculation. More importantly, this study can enhance the awareness of linear algebra in local mathematics education. More improvement can be developed through reflecting the syllabus and curriculum design of local mathematics education.

Research Methodology

The literature review lists the highlights of the syllabus and textbook contents in linear algebra from CIE, Edexcel IALs and HKDSE M2. They are compared in the form of tables in this essay. This study is going to compare the syllabus of mathematics in different qualifications in 2019-2020. The textbook contents are extracted and compared from the UK textbooks, which include Pure Mathematics 1-3 for Cambridge International AS & A Level, Edexcel AS and A Level Modular Mathematics Further Pure Mathematics 1-3, and the HK textbooks, which include New century mathematics extended part second edition (2015 edition) M2A & M2B.

Apart from the literature review, both qualitative and quantitative research methods are going to be used in this study. The participants of this study are separated into 2 groups, the local undergraduates who took HKDSE M2 and those who took UK qualifications mathematics papers before. For qualitative methods, interviews are implemented for 3 randomly selected participants of each group in order to deeply understand their views toward UK qualifications mathematics papers and HKDSE M2. For the quantitative methods, 40 questionnaires are distributed to the local undergraduates who took HKDSE M2 to collect their comments towards linear algebra contents in the local mathematics education. Both questionnaires and interviews can show how the undergraduates respond to mathematics in UK qualifications or HKDSE M2 and assess their learning outcome in linear algebra.

The interviews are conducted online by using the communication application, Zoom. The participants have to answer a few questions to show their views towards the contents of UK qualifications or HKDSE M2 in linear algebra. Also, some documents, comparative tables and extracts from the textbooks about the curriculum of HKDSE M2, CIE or Edexcel IALs in the UK would be prepared and distributed before the interview for students to know the syllabus of both examinations and give their comments on them. Some questions include: please state the difficulty (1-10) of HKDSE M2 or UK qualification mathematics past paper or contents and please provide the comment on the difference about linear algebra between HKDSE M2 and UK qualification math subjects.

The questionnaires are designed and distributed by google form for 40 students who took HKDSE M2 before. They have to answer a few multiple-choice questions or tick the checklist listing their knowledge level of linear algebra. After that, they can provide suggestions to the syllabus and the textbooks of M2 or the local mathematics education.

Overview of Qualification

I. Cambridge International Examination (CIE)

Cambridge International Examination (CIE) is provided by Cambridge Assessment International Education (CAIE) (or Cambridge International), which provides a lot of international educational programs and qualifications for students from primary to high school. The qualifications and examinations of about 10,000 schools in more than 160 countries are provided by CAIE. Therefore, the qualifications of CIE are globally recognized by the universities in the UK, the United States, Germany, New Zealand, Australia, European Union and Canada, like all Ivy League universities, Stanford, Oxford and Cambridge. 55 subjects are available in the CIE.

According to the Cambridge International AS and A Level results statistic, in the core mathematics examination of CIE A level in November 2019, the number of students achieving C or above is more than half of all candidates, the percentage of students achieving B or above is about 37% and that of students achieving A or above is about 22%. 6 papers are included in the subject of core mathematics of CIE, which are 1. Pure Mathematics 1, 2. Pure Mathematics 2, 3. Pure Mathematics 3, 4. Mechanics, 5. Probability & Statistics 1 and 6. Probability & Statistics 2. To get AS level in mathematics in CIE, students have to take either combining paper 1 and 2, combining paper 1 and 4 or combining paper 1 and 5. To get A level in mathematics in CIE, they have to take either paper 1, 3, 4, 5 or paper 1, 3, 5, 6.

In the further mathematics examination of CIE A level in November 2019, around 80% of candidates can achieve C or above, around 55% of candidates can achieve A or above and 32% of them can achieve A*. The CIE of further pure mathematics subjects includes 4 papers, which are Further Pure Mathematics 1, Further Pure Mathematics 2, Further Mechanics and Further

Probability & Statistics. Students have to take either paper 1 and 3 or paper 1 and 4 to get AS level in Further Mathematics. They can also take all 4 papers to get A level instead. Figure 1 below shows the routes through AS & A level mathematics and AS & A level Further Mathematics:

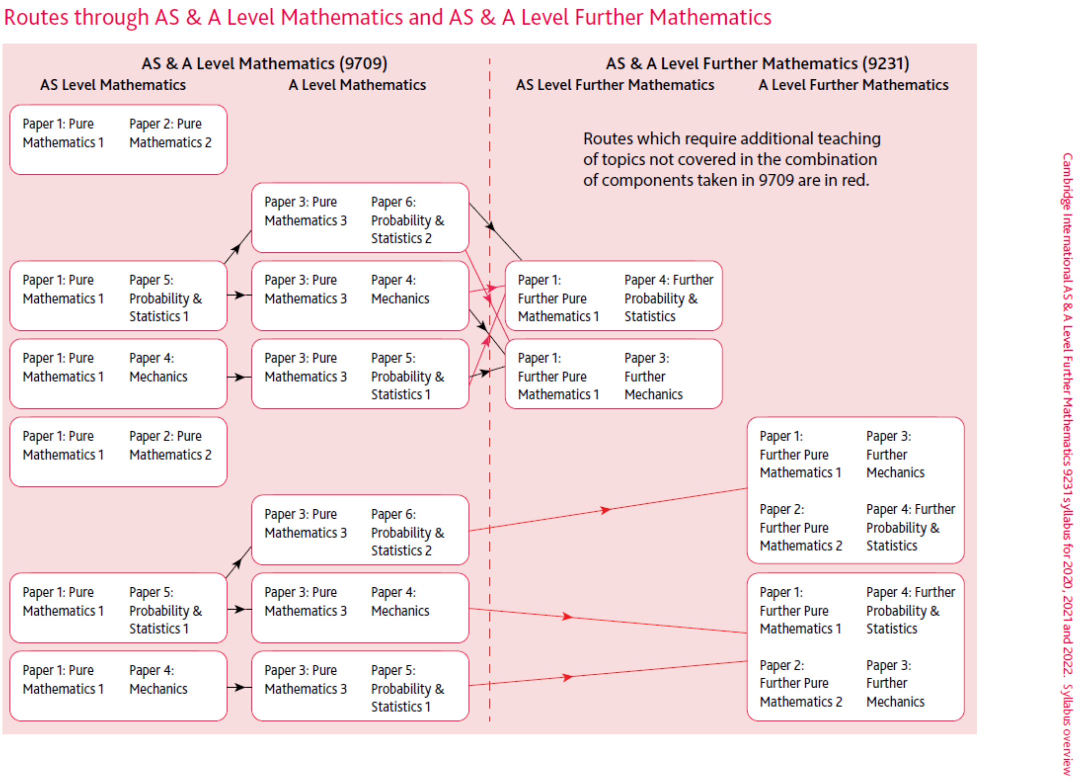


Figure 1. Routes through AS & A level mathematics and AS & A level Further Mathematics

II. Edexcel IALs

Edexcel International Advanced Levels (Edexcel IALs) is a kind of Pearson's Edexcel qualifications for students from foreign countries, which is globally recognized and has a similar standard as Edexcel GCE A Levels. 12 subjects are available for the qualification. The structure and contents of the mathematics subjects of Edexcel IALs are quite similar to that of CIE. Edexcel

IALs also includes pure mathematics (P1-4), further pure mathematics (FP1-3), mechanics (M1-4), probability (S1-3) and decision mathematics (D1) subject papers. In the core mathematics examination of Edexcel IALs in January 2020, there are a total of 1207 candidates. From the grade statistics of Edexcel IALs, around 70% of candidates can achieve B or above, around 50% of candidates can achieve A or above and 30% of them can achieve A* in pure mathematics. In the further mathematics examination of Edexcel IALs in January 2020, there are a total of 277 candidates. Around 70% of candidates can achieve B or above, around 50% of candidates can achieve A or above and 20% of them can achieve A*. Figure 2 below shows a qualification overview of Edexcel IALs:

Qualification	Compulsory units	Optional units
International Advanced Subsidiary in Mathematics	P1, P2	M1, S1, D1
International Advanced Subsidiary in Further Mathematics	FP1	FP2, FP3, M1, M2, M3, S1, S2, S3, D1
International Advanced Subsidiary in Pure Mathematics	P1, P2, FP1	

Pearson Edexcel International Advanced Level

The International Advanced Level in Mathematics, Further Mathematics and Pure Mathematics qualifications each consist of six externally-examined units:

Qualification	Compulsory units	Optional units
International Advanced Level in Mathematics	P1, P2, P3, P4	M1 and S1 or M1 and D1 or M1 and M2 or S1 and D1 or S1 and S2
International Advanced Level in Further Mathematics	FP1 and either FP2 or FP3	FP2, FP3, M1, M2, M3, S1, S2, S3, D1
International Advanced Level in Pure Mathematics	P1, P2, P3, P4, FP1	FP2 or FP3

Figure 2. Qualification overview of Edexcel IALs

III. HKDSE M2

Hong Kong Diploma of Secondary Education Examination (HKDSE) developed by the Hong Kong Examinations and Assessment Authority (HKEAA) replaced the Hong Kong Certificate of Education Examination (HKCEE) and Hong Kong Advanced Level Examination (HKALE) in 2012. This qualification provides the opportunity for local students to enroll in the local universities according to their examination results of both major and elective subjects. Getting level 5** or 5* may be equivalent to getting A* in International General Certificate of Secondary Education (IGCSE) Exam A level, like Edexcel IALs and CIE. From the statistics provided by HKEAA, 54642 candidates entered in HKDSE and 4345 candidates entered in the examination of HKDSE M2 in 2019. In the same year, only 4% of candidates can get level 5**, around 11% of candidates can get level 5*, 23% of candidates can get level 5 and around 60% of candidates can get level 4 or above.

Literature Review

I. Basic of linear algebra

Linear algebra is the core of this study, which is the study of linear equations, linear maps and how they represent through matrices and in the vector spaces. It helps represent and operate the sets of linear equations. For example, the vector can be imagined as a point or an arrow in space. The matrices are another system to represent the scalars or coordinates of the vectors. The linear transformation of a vector or a plane in 2D and 3D planes can be done by the multiplication of the matrices. Linear algebra is fundamental to applications in science, engineering and machine learning. It can be applied to the analysis of rotations in space, the shortest distance, projections, finding the solutions of linear or differential equations and least square fitting (Penney, 2008). Learning linear algebra in high school should be able to help students to have a better transition of the mathematical concepts in linear algebra and the performance in the college so that the awareness of the university student towards this mathematical fields could be raised (Thomas, de Freitas Druck, Huillet, Ju, Nardi, Rasmussen & Xie, 2015). Linear spaces and linear mappings, linear products and coproducts, families in linear spaces, finite dimensions and duality are the topics in linear algebra (Schaffer & World Scientific, 2015).

Several major topics like matrices, the system of linear equations and vectors can be considered to be the focus of the comparison between the CIE and HKDSE M2 contents.

Moreover, some researchers noticed that students may encounter difficulties while learning linear algebra (Stewart & Thomas, 2009). For example, they have difficulties in understanding the concept of eigenvalues and eigenvectors (Stewart & Thomas, 2006), subspace (Wawro, Sweeney & Rabin, 2011) or linear independence (Stewart & Thomas, 2010). According to the comparison between different syllabus and the difficulties encountered by the majority of students, some recommendations on the contents or teaching strategies could be suggested to advance the examination contents and teaching in linear algebra, such as geometric representation, algebraic representation and soliciting examples from students (Bogomolny, 2007).

II. Syllabus of CIE maths papers

From the documents of the syllabus of Cambridge International AS & A Level Mathematics and Further Mathematics published by the CAIE, one of the major topics of linear algebra, vectors, is included in the contents of the Pure Mathematics Paper 3 of CIE, which is the compulsory paper of getting A level in CIE. Also, the topic of vectors is included in CIE Further Pure Mathematics Paper 1 and both the CIE Further Pure Mathematics Paper 1 and Paper 2 include the topic of matrices in the content. Furthermore, some contents in CIE Further Mechanics Paper 3 include the application of the vectors or knowledge in linear algebra, like the motion of a projectile and equilibrium of a rigid body.

In Pure Mathematics Paper 3 of CIE, the students should require the knowledge of using the standard notations for vectors, carry out the basic operation of vectors, using displacement vectors, position vectors and unit vector to calculate the magnitude of a vector, find the intersection of two lines or determine whether they are parallel, express the equation of a line in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and problem solving by using the scalar product of two vectors (Figure 3).

3.7 Vectors	
Candidates should be able to:	Notes and examples
• use standard notations for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$	
• carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms	e.g. '$OABC$ is a parallelogram' is equivalent to $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$. The general form of the ratio theorem is not included, but understanding that the midpoint of AB has position vector $\frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$ is expected.
• calculate the magnitude of a vector, and use unit vectors, displacement vectors and position vectors	In 2 or 3 dimensions.
• understand the significance of all the symbols used when the equation of a straight line is expressed in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, and find the equation of a line, given sufficient information	e.g. finding the equation of a line given the position vector of a point on the line and a direction vector, or the position vectors of two points on the line.
• determine whether two lines are parallel, intersect or are skew, and find the point of intersection of two lines when it exists	Calculation of the shortest distance between two skew lines is not required. Finding the equation of the common perpendicular to two skew lines is also not required.
• use formulae to calculate the scalar product of two vectors, and use scalar products in problems involving lines and points.	e.g. finding the angle between two lines, and finding the foot of the perpendicular from a point to a line; questions may involve 3D objects such as cuboids, tetrahedra (pyramids), etc. Knowledge of the vector product is not required.

Figure 3. Vectors in Pure Mathematics Paper 3 of CIE

In CIE Further Pure Mathematics Paper 1, both matrices and vectors are included. The basic operation of matrix, introduction of zero matrix and identity matrix, finding determinants and inverses of non-singular matrices, representation of geometric transformations in the $x - y$ plane and solving problems involving invariant lines and points are introduced in the topic of matrices. Converting equations of planes into different forms, like $ax + by + xz = d$, $\mathbf{r} \cdot \mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, vector product $\mathbf{a} \times \mathbf{b}$ and all contents of vectors covered in the Pure Mathematics Paper 3 of CIE should be used by the candidates in paper 1 (Figure 4 & 5).

1 Further Pure Mathematics 1

1.4 Matrices

Candidates should be able to:	Notes and examples
• carry out operations of matrix addition, subtraction and multiplication, and recognise the terms zero matrix and identity (or unit) matrix	Including non-square matrices. Matrices will have at most 3 rows and columns.
• recall the meaning of the terms 'singular' and 'non-singular' as applied to square matrices and, for 2×2 and 3×3 matrices, evaluate determinants and find inverses of non-singular matrices	The notations $\det \mathbf{M}$ for the determinant of a matrix $\mathbf{M}$, and $\mathbf{I}$ for the identity matrix, will be used.
• understand and use the result, for non-singular matrices, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$	Extension to the product of more than two matrices may be required.
• understand the use of 2×2 matrices to represent certain geometric transformations in the x-y plane, in particular – understand the relationship between the transformations represented by $\mathbf{A}$ and $\mathbf{A}^{-1}$ – recognise that the matrix product $\mathbf{AB}$ represents the transformation that results from the transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$ – recall how the area scale factor of a transformation is related to the determinant of the corresponding matrix – find the matrix that represents a given transformation or sequence of transformations	Understanding of the terms 'rotation', 'reflection', 'enlargement', 'stretch' and 'shear' for 2D transformations will be required. Other 2D transformations may be included, but no particular knowledge of them is expected.
• understand the meaning of 'invariant' as applied to points and lines in the context of transformations represented by matrices, and solve simple problems involving invariant points and invariant lines.	e.g. to locate the invariant points of the transformation represented by $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$, or to find the invariant lines through the origin for $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, or to show that any line with gradient 1 is invariant for $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

Figure 4. Matrices in Further Pure Mathematics Paper 1 of CIE

1.6 Vectors

Candidates should be able to: | Notes and examples

- use the equation of a plane in any of the forms $ax + by + cz = d$ or $\mathbf{r}.\mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ and convert equations of planes from one form to another as necessary in solving problems
- recall that the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors can be expressed either as $|\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector, or in component form as $(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$
- use equations of lines and planes, together with scalar and vector products where appropriate, to solve problems concerning distances, angles and intersections, including
  - determining whether a line lies in a plane, is parallel to a plane or intersects a plane, and finding the point of intersection of a line and a plane when it exists
  - finding the foot of the perpendicular from a point to a plane
  - finding the angle between a line and a plane, and the angle between two planes
  - finding an equation for the line of intersection of two planes
  - calculating the shortest distance between two skew lines
  - finding an equation for the common perpendicular to two skew lines.

Figure 5. Vectors in Further Pure Mathematics Paper 1 of CIE

Further usage of matrices is introduced in Further Pure Mathematics Paper 2, which includes consistency or inconsistency of 3 linear simultaneous equations, characteristic equation, eigenvalue and eigenvector of 2×2 and 3×3 matrices (Figure 6).

2.2 Matrices	
Candidates should be able to:	Notes and examples
• formulate a problem involving the solution of 3 linear simultaneous equations in 3 unknowns as a problem involving the solution of a matrix equation, or vice versa	
• understand the cases that may arise concerning the consistency or inconsistency of 3 linear simultaneous equations, relate them to the singularity or otherwise of the corresponding matrix, solve consistent systems, and interpret geometrically in terms of lines and planes	e.g. three planes meeting in a common point, or in a common line, or having no common points.
• understand the terms 'characteristic equation', 'eigenvalue' and 'eigenvector', as applied to square matrices	Including use of the definition $\mathbf{Ae} = \lambda\mathbf{e}$ to prove simple properties, e.g. that λ^n is an eigenvalue of $\mathbf{A}^n$.
• find eigenvalues and eigenvectors of 2×2 and 3×3 matrices	Restricted to cases where the eigenvalues are real and distinct.
• express a square matrix in the form $\mathbf{QDQ}^{-1}$, where $\mathbf{D}$ is a diagonal matrix of eigenvalues and $\mathbf{Q}$ is a matrix whose columns are eigenvectors, and use this expression	e.g. in calculating powers of 2×2 or 3×3 matrices.
• use the fact that a square matrix satisfies its own characteristic equation.	e.g. in finding successive powers of a matrix or finding an inverse matrix; restricted to 2×2 or 3×3 matrices only.

Figure 6. Matrices in Further Pure Mathematics Paper 2 of CIE

III. Syllabus of Edexcel IALS math papers

From the International Advanced Level Mathematics/ Further Mathematics/ Pure Mathematics specification published by Pearson Edexcel, the linear algebra contents like vectors and matrices are included in the syllabus of Pure Mathematics 4 (P4), Further Mathematics 1 and 3 (FP1 and FP3) and Mechanics 1 (M1) of Edexcel IALs.

In P4, basic of vectors, magnitude of vector, position vectors, vector basic operation, the distance between two points, vector equation of lines and scalar product are introduced (Figure 7).

7. Vectors		
7.1	Vectors in two and three dimensions.	
7.2	Magnitude of a vector.	Students should be able to find a unit vector in the direction of **a**, and be familiar with $\lvert \mathbf{a} \rvert$.
7.3	Algebraic operations of vector addition and multiplication by scalars, and their geometrical interpretations.	
7.4	Position vectors.	$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

What students need to learn:		Guidance
7. Vectors *continued*		
7.5	The distance between two points.	The distance d between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$
7.6	Vector equations of lines.	To include the forms $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + t(\mathbf{d} - \mathbf{c})$ Conditions for two lines to be parallel, intersecting or skew.
7.7	The scalar product. Its use for calculating the angle between two lines.	Students should know that for $\overrightarrow{OA} = \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\overrightarrow{OB} = \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $\cos \angle AOB = \dfrac{\mathbf{a}.\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ Students should know that if $\mathbf{a}.\mathbf{b} = 0$, and **a** and **b** are non-zero vectors, then **a** and **b** are perpendicular.

Figure 7. Vectors in Pure Mathematics 4 of Edexcel IALs

In FP1, matrix algebra integration covers the basic operation of matrices, determinants and inverse of 2×2 matrices. The topic of transformations using matrices also covers the linear transformation of column vectors in 2 dimensions, representation geometrical transformations by 2×2 matrices, combinations of transformations and the inverse of transformation (Figure 8).

What students need to learn:		Guidance
5. Matrix algebra integration		
5.1	Addition and subtraction of matrices.	
5.2	Multiplication of a matrix by a scalar.	
5.3	Products of matrices.	
5.4	Evaluation of 2×2 determinants.	Singular and non-singular matrices.
5.5	Inverse of 2×2 matrices.	Use of the relation $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
6. Transformations using matrices		
6.1	Linear transformations of column vectors in two dimensions and their matrix representation.	The transformation represented by **AB** is the transformation represented by **B** followed by the transformation represented by **A**.
6.2	Applications of 2×2 matrices to represent geometrical transformations.	Identification and use of the matrix representation of single transformations from: reflection in coordinate axes and lines $y = \pm x$, rotation through any angle about (0, 0), stretches parallel to the x-axis and y-axis, and enlargement about centre (0, 0), with scale factor k, $(k \neq 0)$, where $k \in \mathbb{R}$.
6.3	Combinations of transformations.	Identification and use of the matrix representation of combined transformations.
6.4	The inverse (when it exists) of a given transformation or combination of transformations.	Idea of the determinant as an area scale factor in transformations.

Figure 8. Matrices in Further Mathematics 1 of Edexcel IALs

In FP3, vector product $\mathbf{a} \times \mathbf{b}$, triple scalar product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, line equation in the form of $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$ and plane equation in the forms of $\mathbf{r} \cdot \mathbf{n} = p$, $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$ are introduced (Figure 9). Further matrix algebra covers transpose of a matrix, inverse of 3×3 matrices, eigenvalues and eigenvectors of 2×2 and 3×3 matrices and reduction of symmetric matrices to diagonal form (Figure 10).

5. Vectors		
5.1	The vector product **a** × **b** and the triple scalar product **a** . **b** × **c**.	The interpretation of | **a** × **b** | as an area and **a** . **b** × **c** as a volume.
5.2	Use of vectors in problems involving points, lines and planes. The equation of a line in the form (**r** − **a**) × **b** = **0**.	Students may be required to use equivalent cartesian forms also. Applications to include (i) distance from a point to a plane, (ii) line of intersection of two planes, (iii) shortest distance between two skew lines.
5.3	The equation of a plane in the forms **r**.**n** = *p*, **r** = **a** + *s***b** + *t***c**.	Students may be required to use equivalent cartesian forms also.

Figure 9. Vectors in Further Mathematics 3 of Edexcel IALs

What students need to learn:		Guidance
6. Further matrix algebra		
6.1	Linear transformations of column vectors in two and three dimensions and their matrix representation.	Extension of work from FP1 to 3 dimensions.
6.2	Combination of transformations. Products of matrices.	The transformation represented by **AB** is the transformation represented by **B** followed by the transformation represented by **A**.
6.3	Transpose of a matrix.	Use of the relation $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.
6.4	Evaluation of 3 × 3 determinants.	Singular and non-singular matrices.
6.5	Inverse of 3 × 3 matrices.	Use of the relation $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
6.6	The inverse (when it exists) of a given transformation or combination of transformations.	
6.7	Eigenvalues and eigenvectors of 2 × 2 and 3 × 3 matrices.	Normalised vectors may be required.
6.8	Reduction of symmetric matrices to diagonal form.	Students should be able to find an orthogonal matrix **P** such that $\mathbf{P}^T\mathbf{AP}$ is diagonal.

Figure 10. Further matrix algebra in Further Mathematics 3 of Edexcel IALs

In M1, direction and magnitude of vectors are applied to displacements, accelerations, forces and velocities in a plane (Figure 11).

2. Vectors in mechanics		
2.1	Magnitude and direction of a vector. Resultant of vectors may also be required.	Students may be required to resolve a vector into two components or use a vector diagram. Questions may be set involving the unit vectors **i** and **j**.
2.2	Application of vectors to displacements, velocities, accelerations and forces in a plane.	Use of velocity = $\frac{\text{change of displacement}}{\text{time}}$ in the case of constant velocity, and of acceleration = $\frac{\text{change of velocity}}{\text{time}}$ in the case of constant acceleration, will be required.

Figure 11. Vectors in mechanics in Mechanics 1of Edexcel IALs

IV. Syllabus of HKDSE M2

From Mathematics Curriculum and Assessment Guide (Secondary 4 -6) published by the Curriculum and Assessment Council and HKEAA and Explanatory notes to senior secondary mathematics curriculum: Module 2 (Algebra and Calculus) published by the Education Bureau, the part of algebra area in HKDSE M2 contents includes the topics of determinants, matrices, systems of linear equations, introduction to vectors, scalar product or vector product, and application of vectors.

Determinants introduce its definition of 2×2 and 3×3 matrices and some geometric uses of determinants, like the area of a triangle with given coordinates of points on the rectangular coordinate plane. Matrices introduce the basic properties and operation of matrices, zero matrix, identity matrix, transpose of a matrix, square matrix and properties of inverse of square matrices of order 2 and 3. Using adjoint matrix and elementary row operations to determine the invertibility of square matrix and find its inverse is also covered in this topic. Cramer's rule, inverse matrices and Gaussian elimination are introduced in solving systems of the linear equation. The properties of homogeneous and consistency and the theorem that nontrivial solutions of a system of the homogeneous linear equation can be found if and only if there is a singular coefficient matrix are also included in this topic.

Introduction of vectors covers the concept, operations, properties of vectors and its representation in the rectangular coordinate systems. Scalar product and vector product covers the properties and definition of the scalar product of vectors and vector product of vectors in $\mathbf{R}^3$, determinant form of the vector product and the geometric meanings of scalar product and vector product. Application of vectors covers using vector properties to solve problems with parallelism and orthogonality, line segment division and projection of a vector onto another vector, finding angles between 2 vectors and the area of triangle or parallelogram with scalar product and vector product. Figures below (Figure 12 - 17) shows the syllabus of different topics of linear algebra in HKDSE M2:

Algebra			
12. Determinants	12.1 recognise the concept of determinants of order 2 and order 3	2	Students are required to recognise the notations: $\|A\|$ and $\det A$.
13. Matrices	13.1 understand the concept, operations and properties of matrices	10	The addition, scalar multiplication and multiplication of matrices are required. The properties include: • $A+B=B+A$ • $A+(B+C)=(A+B)+C$ • $(\lambda+\mu)A=\lambda A+\mu A$ • $\lambda(A+B)=\lambda A+\lambda B$ • $A(BC)=(AB)C$ • $A(B+C)=AB+AC$

Figure 12. Determinant and matrices in HKDSE M2

Learning Unit	**Learning Objective**	**Time**	**Remarks**
			• $(A+B)C=AC+BC$ • $(\lambda A)(\mu B)=(\lambda\mu)AB$ • $\|AB\|=\|A\|\|B\|$
	13.2 understand the concept, operations and properties of inverses of square matrices of order 2 and order 3		The properties include: • the inverse of A is unique • $(A^{-1})^{-1}=A$ • $(\lambda A)^{-1}=\lambda^{-1}A^{-1}$ • $(A^n)^{-1}=(A^{-1})^n$ • $(A^T)^{-1}=(A^{-1})^T$ • $\|A^{-1}\|=\|A\|^{-1}$ • $(AB)^{-1}=B^{-1}A^{-1}$ where A and B are invertible matrices and λ is a non-zero scalar.

Figure 13. Matrices in HKDSE M2

Learning Unit	Learning Objective	Time	Remarks
14. Systems of linear equations	14.1 solve the systems of linear equations in two and three variables by Cramer's rule, inverse matrices and Gaussian elimination	6	The following theorem is required: A system of homogeneous linear equations has nontrivial solutions if and only if the coefficient matrix is singular
15. Introduction to vectors	15.1 understand the concepts of vectors and scalars	5	The concepts of magnitudes of vectors, zero vector and unit vectors are required. Students are required to recognise some common notations of vectors in printed form (including **a** and $\overrightarrow{AB}$) and in written form (including $\vec{a}$, $\overrightarrow{AB}$ and $\underset{\sim}{a}$) ; and some notations for magnitude (including $\lvert\mathbf{a}\rvert$ and $\lvert\vec{a}\rvert$).
	15.2 understand the operations and properties of vectors		The addition, subtraction and scalar multiplication of vectors are required. The properties include: • $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ • $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$ • $\mathbf{a}+\mathbf{0}=\mathbf{a}$ • $0\,\mathbf{a}=\mathbf{0}$

Figure 14. System of lineaer equations and introduction to vectors in HKDSE M2

Learning Unit	Learning Objective	Time	Remarks
			• $\lambda(\mu\mathbf{a})=(\lambda\mu)\mathbf{a}$ • $(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}$ • $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$ • If $\alpha\mathbf{a}+\beta\mathbf{b}=\alpha_1\mathbf{a}+\beta_1\mathbf{b}$ (**a** and **b** are non-zero and are not parallel to each other), then $\alpha=\alpha_1$ and $\beta=\beta_1$
	15.3 understand the representation of a vector in the rectangular coordinate system		The formulae that students are required to use include: • $\lvert\overrightarrow{OP}\rvert=\sqrt{x^2+y^2+z^2}$ in $\mathbf{R}^3$ • $\sin\theta=\frac{y}{\sqrt{x^2+y^2}}$ and $\cos\theta=\frac{x}{\sqrt{x^2+y^2}}$ in $\mathbf{R}^2$ The representation of vectors in the rectangular coordinate system can be used to discuss those properties listed in the Remarks against Learning Objective 15.2. The concept of direction cosines is **not** required.

Figure 15. Introduction to vectors in HKDSE M2

Learning Unit	Learning Objective	Time	Remarks
16. Scalar product and vector product	16.1 understand the definition and properties of the scalar product (dot product) of vectors	5	The properties include: • $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ • $\mathbf{a}\cdot(\lambda\mathbf{b})=\lambda(\mathbf{a}\cdot\mathbf{b})$ • $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ • $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert^2\geq 0$ • $\mathbf{a}\cdot\mathbf{a}=0$ if and only if $\mathbf{a}=\mathbf{0}$ • $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\geq\lvert\mathbf{a}\cdot\mathbf{b}\rvert$ • $\lvert\mathbf{a}-\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2-2(\mathbf{a}\cdot\mathbf{b})$
	16.2 understand the definition and properties of the vector product (cross product) of vectors in $\mathbf{R}^3$		The properties include: • $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ • $\mathbf{b}\times\mathbf{a}=-(\mathbf{a}\times\mathbf{b})$ • $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$ • $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ • $(\lambda\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\lambda\mathbf{b})=\lambda(\mathbf{a}\times\mathbf{b})$ • $\lvert\mathbf{a}\times\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2\lvert\mathbf{b}\rvert^2-(\mathbf{a}\cdot\mathbf{b})^2$

Figure 16. Introduction to vectors in HKDSE M2

Learning Unit	Learning Objective	Time	Remarks
17. Applications of vectors	17.1 understand the applications of vectors	6	Division of a line segment, parallelism and orthogonality are required. Finding angles between two vectors, the projection of a vector onto another vector and the area of a triangle are required.
	Subtotal in hours	34	

Figure 17. Introduction to vectors in HKDSE M2

V. Syllabus Comparison

By comparing with the syllabus of the CIE Mathematics papers, Edexcel IALs Mathematics papers and HKDSE M2, the topics relating to linear algebra are collected and sorted out in the following figures (Figure 18 &19). There are still so many methods to compare different syllabus and curriculum (Chong, Chang, Kim & Kwon…, 2016).

			CIE			Edexcel IALs			HKDSE
Comparison between CIE, Edexcel IALs & HKDSE M2 syllabus	Syllabus content	P3	FP1	FP2	P4	FP1	FP3	M1	M2
Vectors	standard notations	✓			✓				✓
	basic operation of vectors	✓			✓				✓
	representation in rectangular coordinate system								✓
	displacement vectors	✓							
	position vectors	✓			✓				
	unit vector	✓							
	magnitude of a vector	✓			✓				
	find the intersection of two lines or determine whether they are parallel	✓							
	express the equation of a line in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$	✓			✓				
	using scalar product of two vectors	✓			✓				✓
	using vector product of two vectors				✓		✓		✓
	determinant form of the vector product								✓
	geometric meanings of scalar product and vector product								✓
	triple scalar product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$						✓		
	line equation in the form of $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$						✓		
	Converting equations of planes into different forms: $\mathbf{r} \cdot \mathbf{n} = p$, $ax + by + xz = d$, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$		✓				✓		
	distance between two points				✓				
	vector application to displacements, accelerations, forces and velocities in a plane							✓	

Figure 18. Syllabus comparison between CIE, Edexcel IALs & HKDSE M2

		CIE			Edexcel IALs				HKDSE
		P3	FP1	FP2	P4	FP1	FP3	M1	M2
Matrices	geometric uses of determinants								✓
	basic operation of matrices					✓			✓
	zero matrix		✓						✓
	identity matrix		✓						✓
	determinants of non-singular matrices		✓			✓			
	inverses of non-singular matrices		✓			✓	✓		✓
	transpose of a matrix						✓		✓
	Using adjoint matrix and elementary row operations to determine the invertibility of square matrix and find its inverse								✓
	representation of geometric transformations in the $x-y$ plane		✓			✓			
	combinations of transformations					✓			
	the inverse of transformation					✓			
	solving problems involving invariant lines and points		✓						
	consistency or inconsistency of 3 linear simultaneous equations			✓					✓
	characteristic equation			✓					
	eigenvalue and eigenvector of 2×2 and 3×3 matrices			✓			✓		
	reduction of symmetric matrices to diagonal form						✓		
	Cramer's rule								✓
	Gaussian elimination								✓
	The properties of homogeneous								✓

Figure 19. Syllabus comparison between CIE, Edexcel IALs & HKDSE M2

Since the structure and the syllabus content relating to linear algebra of CIE math paper are quite similar to Edexcel IALs math paper, more emphasis would be put on the difference between the math syllabus of 2 UK qualifications with HKDSE M2.

Some vector contents are mentioned in the UK qualifications math syllabus but not appeared in the HKDSE M2 syllabus, for example, displacement vectors, position vectors, unit vector, magnitude of a vector, finding the intersection of two lines or determine whether they are parallel, expressing the equation of a line in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, triple scalar product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, expressing line equation in the form of $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$ and plane equation in the forms of $ax + by + xz = d$, $\mathbf{r} \cdot \mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, distance between two points and vector application to displacements, accelerations, forces and velocities in a plane . The matrix contents mentioned in the UK qualifications math syllabus but not appeared in the HKDSE M2 syllabus are the representation of geometric transformations in the $x - y$ plane, combinations of transformations and the inverse of transformation, solving problems involving invariant lines and points, characteristic equation, eigenvalue and eigenvector of 2×2 and 3×3 matrices, reduction of symmetric matrices to diagonal form.

The vector contents mentioned in the HKDSE M2 syllabus but not appeared in the UK qualifications maths syllabus are representation in the rectangular coordinate system, determinant form of the vector product and geometric meanings of scalar product and vector product. The matrix contents mentioned in the HKDSE M2 syllabus but not appeared in the UK qualifications maths syllabus are geometric uses of determinants, using adjoint matrix and elementary row operations to determine the invertibility of square matrix and find its inverse, Cramer's rule, Gaussian elimination, and homogeneous property of matrix. The topic of Systems of linear equations is not covered in the UK qualifications math syllabus.

In addition, the compulsory CIE A level and Edexcel IALs pure mathematics subject covers the topic of vectors but there is no content about linear algebra in the compulsory part of HKDSE mathematics papers. Moreover, vector is applied in one of the topics in the Mechanics syllabus, Vectors in mechanics. Therefore, UK qualifications math syllabus requires students to have a higher level of knowledge in linear algebra than HKDSE M2 syllabus does. Linear algebra is more common in the UK than in Hong Kong since the students who study compulsory math and mechanics are required to learn vectors.

The guided learning hours for the algebra part of HKDSE M2 is 34 hours, 25% out of 125 learning hours for the whole module. The guided lesson hours for the compulsory part with M2 is 375 hours. On the other hand, students are designed to have 180 guided learning hours for Cambridge International AS Level and 360 hours for A Level. The lesson time arrangement of Edexcel IALs is similar to that of CIE. The detailed learning hour for pure and further mathematics is not stated.

The syllabus may not be accurate enough to cover all the contents of vectors, matrix or linear algebra. Some contents may also be overstated. So, the above summary may be incomplete. It could be more accurate after comparing the textbook content of HKDSE M2 and UK qualification math textbooks.

VI. UK textbooks content in linear algebra

Since the structure and the syllabus of CIE A level and Edexcel IALs maths qualifications are very similar, the textbook Pure Mathematics 1-3 for Cambridge International AS & A Level, Edexcel AS and A Level Modular Mathematics Further Pure Mathematics 1-3 and Mechanic 1 would be extracted to represent the textbook content in linear algebra in the UK and compare with Hong Kong textbook. They are published by OXFORD University Press in 2015 and Pearson Education Limited in 2016 respectively.

The first chapter of linear algebra in Pure mathematics is vector, which is about 11% of the whole P3 textbook. There are some contents of linear algebra covered in the textbook have not been mentioned in the CIE A level syllabus, such as negative vector, finding the intersection of two lines and the distance from a point to a line by using line equation.

The second chapter of linear algebra is matrix algebra in Further mathematics 1, which is about 30% of the whole textbook. It covers determining the invertibility of square matrix and find its inverse, using determinants of a matrix to determine the area scalar factor of the transformation and solving system of linear equations by inverse matrix method. The geometric representations of linear transformation, combination of transformation and inverse of transformation are illustrated in the textbooks.

The third chapter is vectors in Further mathematics 3, which is about 15% of the whole textbook. It covers vector projection, properties of vector products, vector products and areas of parallelogram and triangles, volume of parallelepiped and tetrahedron, triple scalar product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, general Cartesian equation of a straight line, angles between line and plane, angles between 2 planes and the shortest distance between 2 skew lines with equations. The fourth chapter is further matrix algebra in Further mathematics 3, which is about 17% of the whole textbook. It covers properties of determinants, singular matrix/ non-singular matrix, properties of inverse, minor of a matrix, characteristic equation of matrix, normalized eigenvector, orthogonal matrix and eigenvectors and diagonal matrix.

The final chapter is vectors in Mechanic 1, which is about 10% of the whole textbook. The contents cover the velocity of a particle as a vector, solving problems involving velocity and time using vectors and solving problems about force. The other contents are similar to vectors in the Pure Mathematics textbook.

VII. Hong Kong textbooks content in linear algebra

New century mathematics extended part second edition (2015 edition) M2A & M2B, published in 2015 by Oxford University Press (China), would be used to represent the textbook content in linear algebra in Hong Kong.

The first topic in linear algebra in the Hong Kong textbook is matrices, which is 17% of the textbook. It explains more on the properties of the inverse, determinants and scalar products.

The second topic in linear algebra is solving systems of linear equations, which is 12% of the textbook. The finding is similar to the syllabus before.

The third topic in linear algebra is vectors, which is 15% of the textbook. The scalar multiplication of vectors and section formula used in vector are the contents in the textbooks but not mentioned in the syllabus.

The final topic in linear algebra is scalar product and vector products, which is 11% of the textbook. It covers vector projection, properties of vector products, vector products and areas of parallelogram and triangles and volume of a parallelepiped.

VIII. Textbook Content Comparison

After comparing the textbooks in the UK with those in Hong Kong, more teaching contents and differences can be found. The comparative table is modified and the highlighted items have been modified after comparing textbooks (Figure 20 & 21).

For vectors, the UK ——textbooks teach students to find the distance by using the line equation in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, displacement vectors, find the intersection of two lines, find the distance from a point to a line, volume of tetrahedron, line equation in the form of $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$, general Cartesian equation of a straight line, converting equations of planes into different forms: $ax + by + xz = d$, $\mathbf{r} \cdot \mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, angles between line and plane, angles between 2 planes, shortest distance between 2 skew lines with equations, vector application to displacements, accelerations, forces and velocities in a plane, the velocity of a particle as a vector, solving problems involving velocity, time and force using vectors while the Hong Kong textbooks do not.

For matrix, they can teach students to describe linear transformation by using matrices, such as rotations, translations, reflections and enlargements, combinations of transformations, the inverse of transformation, using determinants of a matrix to determine the area scalar factor of the transformation, singular matrix/ non-singular matrix, property of transpose of a matrix, minor of a matrix, solving problems involving invariant lines and points, characteristic equation of matrix, eigenvalue and eigenvector of 2×2 and 3×3 matrices, normalized eigenvector, orthogonal matrix and eigenvectors, diagonal matrix and reduction of symmetric matrices to diagonal form. These have not been mentioned in the Hong Kong textbooks. The contents in linear algebra extracted from the UK textbooks but not included in the Hong Kong textbooks is summarized in the table below:

Topics extracted from the UK textbook (not included in the Hong Kong textbooks)	
Vectors	Matrix
find the distance by using the line equation in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$	describe linear transformation by using matrices, such as rotations: about (0,0) of angles that are multiplication of 45° enlargements: centre (0,0) of scalar factor $k(k \neq 0, k \in \mathbf{R})$ reflections: in coordinate axes or the lines $y = \pm x$ identity: the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called $\mathbf{I}$ and does not carry out any transformation
displacement vectors $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} or\ 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$	the inverse of transformation
find the intersection of two lines which are given by vector equations $\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + t\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ respectively	solving problems involving invariant lines and points
find the distance from a point to a line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$	singular matrix/ non-singular matrix If $det(\mathbf{A}) = 0$, then $\mathbf{A}$ is a singular matrix and $\mathbf{A}^{-1}$ cannot be found. If $det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is a non-singular matrix and $\mathbf{A}^{-1}$ exists.
volume of tetrahedron $\left\lvert \frac{1}{6}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \right\rvert$	property of transpose of a matrix

line equation in the form of $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$	minor of a matrix
general Cartesian equation of a straight line is $$\frac{(x-x_1)}{l}=\frac{(y-y_1)}{m}=\frac{(z-z_1)}{n}=\lambda,$$ where the lines passes through the point (x_1,y_1,z_1), has direction ratios $l{:}m{:}n$, and where λ is a parameter.	using determinants of a matrix to determine the area scalar factor of the transformation $$\textbf{Area of image}$$ $$=\textbf{Area of object}\times\lvert \boldsymbol{det}(\mathbf{M})\rvert$$
converting equations of planes into different forms: $ax+by+xz=d$, $\mathbf{r}\cdot\mathbf{n}=p$ or $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$	characteristic equation of $\mathbf{A}$ is the equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$.
angles between line and plane $$\sin\theta=\left\lvert\frac{\mathbf{b}\cdot\mathbf{n}}{\lvert\mathbf{b}\rvert\lvert\mathbf{n}\rvert}\right\rvert$$	An eigenvector of a matrix $\mathbf{A}$ is a non-zero column vector $\mathbf{x}$ which satisfies the equation $\mathbf{Ax}=\lambda\mathbf{x}$, where λ is a scalar which is the corresponding eigenvalue of the matrix.
angles between 2 planes $$\cos\theta=\left\lvert\frac{\mathbf{n_1}\cdot\mathbf{n_2}}{\lvert\mathbf{n_1}\rvert\lvert\mathbf{n_2}\rvert}\right\rvert$$	If $\begin{pmatrix}a\\b\\c\end{pmatrix}$ is an eigenvector of a matrix, then the corresponding normalised eigenvector is $\begin{pmatrix}\frac{a}{\sqrt{(a^2+b^2+c^2)}}\\\frac{b}{\sqrt{(a^2+b^2+c^2)}}\\\frac{c}{\sqrt{(a^2+b^2+c^2)}}\end{pmatrix}$.
shortest distance between 2 skew lines with equations $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and $\mathbf{r}=\mathbf{c}+\mu\mathbf{d}$, where λ and μ are scalars, is given by the formula $$\left\lvert\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{\lvert\mathbf{b}\times\mathbf{d}\rvert}\right\rvert$$	If $\mathbf{M}$ is an orthogonal matrix consisting of the normalized column vectors $\mathbf{x}_1,\mathbf{x}_2$ and $\mathbf{x}_3$, then $\mathbf{x}_1\cdot\mathbf{x}_2=\mathbf{x}_2\cdot\mathbf{x}_3=\mathbf{x}_3\cdot\mathbf{x}_1=0$.
vector application to displacements, accelerations, forces	diagonal matrix: $\begin{pmatrix}a&0\\0&b\end{pmatrix}$ and $\begin{pmatrix}d&0&0\\0&e&0\\0&0&f\end{pmatrix}$.

If a particle starts from the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, then its displacement from its initial position at time t is $\mathbf{v}t$ and its position vector $\mathbf{r}$ is given by $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t.$	reduction of symmetric matrices to diagonal form: When symmetric matrix $\mathbf{A}$ is reduced to a diagonal matrix $\mathbf{D}$, the elements on the diagonal are the eigenvalue of $\mathbf{A}$.
If a particle with initial velocity $\mathbf{u}$ moves with constant acceleration $\mathbf{a}$ then its velocity, $\mathbf{v}$, at time t is given by $\mathbf{v} = \mathbf{u} + \mathbf{a}t.$	
The force causes the particle to accelerate: $\mathbf{F} = m\mathbf{a}$, where m is the mass of the particle.	

Comparison between CIE, Edexcel IALs & HKDSE M2 syllabus and textbook content									
	Compare items	CIE			Edexcel IALs				HKDSE
		P3	FP1	FP2	P4	FP1	FP3	M1	M2
	standard notations	✓			✓				✓
	basic operation of vectors	✓			✓				✓
	representation in rectangular coordinate system	✓							✓
	displacement vectors	✓							
	position vectors	✓			✓				✓
	negative vector*	✓							✓
	unit vector	✓							✓
	equality of vectors*	✓							✓
	magnitude of a vector*	✓			✓				✓
	determine whether 2 lines are parallel*	✓							✓
	find the intersection of two lines*	✓							
	angles between 2 vectors and orthogonality	✓							✓
	express the equation of a line in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$	✓			✓				
	find the distance from a point to a line*	✓							
	property of scalar products								✓
	scalar multiplication of vectors								✓
	section formula used in vector								✓
	using scalar product of two vectors	✓			✓				✓
Vectors	using vector product of two vectors				✓		✓		✓
	vector projection						✓		✓
	properties of vector products						✓		✓
	determinant form of the vector product						✓		✓
	vector products and areas of parallelogram and triangles						✓		✓
	geometric meanings of scalar product and vector product						✓		✓
	volume of parallelepiped						✓		✓
	volume of tetrahedron						✓		
	triple scalar product $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$						✓		✓
	line equation in the form of $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$						✓		
	general Cartesian equation of a straight line						✓		
	Converting equations of planes into different forms: $\mathbf{r} \cdot \mathbf{n} = p$ $ax + by + xz = d$ $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$		✓				✓		
	Angles between line and plane						✓		
	Angles between 2 planes						✓		
	shortest distance between 2 skew lines with equations						✓		
	vector application to displacements, accelerations, forces and velocities in a plane							✓	
	the velocity of a particle as a vector							✓	
	solving problems involving velocity and time using vectors							✓	
	solving problems about force							✓	

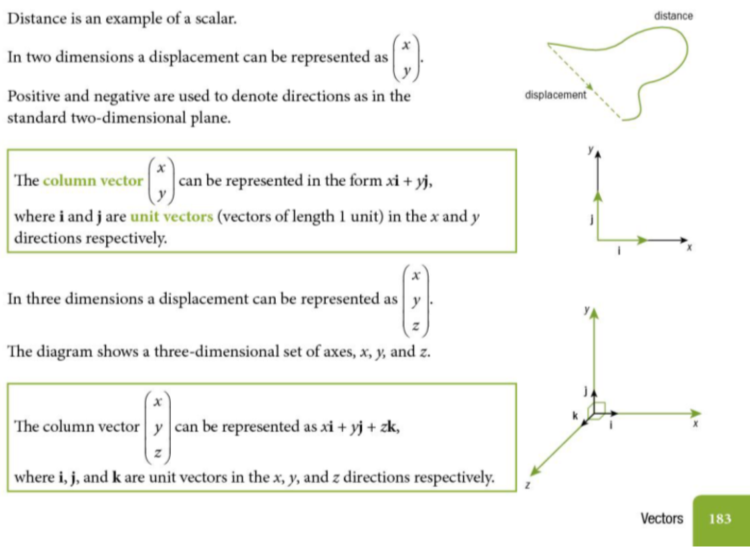
Figure 20. Syllabus and textbook content comparison between CIE, Edexcel IALs and HKDSE M2

			CIE			Edexcel IALs				HKDSE
41				CIE			Edexcel IALs			HKDSE
42			P3	FP1	FP2	P4	FP1	FP3	M1	M2
43	Matrices	geometric uses of determinants								✓
44		basic operation of matrices					✓			✓
45		zero matrix		✓				✓		✓
46		identity matrix		✓				✓		✓
47		determinants of non-singular matrices		✓			✓			✓
48		properties of determinants								✓
49		singular matrix/ non-singular matrix						✓		
50		inverses of non-singular matrices		✓			✓	✓		✓
51		properties of inverse						✓		✓
52		transpose of a matrix						✓		✓
53		property of transpose of a matrix						✓		
54		expanding a determinant						✓		✓
55		minor of a matrix						✓		
56		Using adjoint matrix and elementary row operations to determine the invertibility of square matrix and find its inverse					✓			✓
57		representation of geometric transformations in the $x-y$ plane		✓			✓			
58		combinations of transformations					✓			
59		the inverse of transformation					✓			
60		using determinants of a matrix to determine the area scalar factor of the transformation					✓			
61		solving problems involving invariant lines and points		✓						
62		consistency or inconsistency of 3 linear simultaneous equations			✓					✓
63		characteristic equation of matrix			✓			✓		
64		eigenvalue and eigenvector of 2×2 and 3×3 matrices			✓			✓		
65		normalised eigenvector						✓		
66		orthogonal matrix/ eigenvectors						✓		
67		diagonal matrix						✓		
68		reduction of symmetric matrices to diagonal form						✓		
69		Solving system of linear equations by inverse matrix method					✓			✓
70		Solving system of linear equitons by Cramer's rule								✓
71		Solving system of linear equitons by Gaussian elimination								✓
72		The properties of homogeneous								✓
73										

Figure 21. Syllabus and textbook content comparison between CIE, Edexcel IALs and HKDSE M2

Here are some examples of differences between UK and HK textbooks:

1. UK textbooks denote vectors in both matrix form and algebra form but Hong Kong textbooks denote vectors in algebra form only.


Distance is an example of a scalar.

In two dimensions a displacement can be represented as $\begin{pmatrix} x \\ y \end{pmatrix}$.

Positive and negative are used to denote directions as in the standard two-dimensional plane.

The **column vector** $\begin{pmatrix} x \\ y \end{pmatrix}$ can be represented in the form $x\mathbf{i} + y\mathbf{j}$, where **i** and **j** are **unit vectors** (vectors of length 1 unit) in the x and y directions respectively.

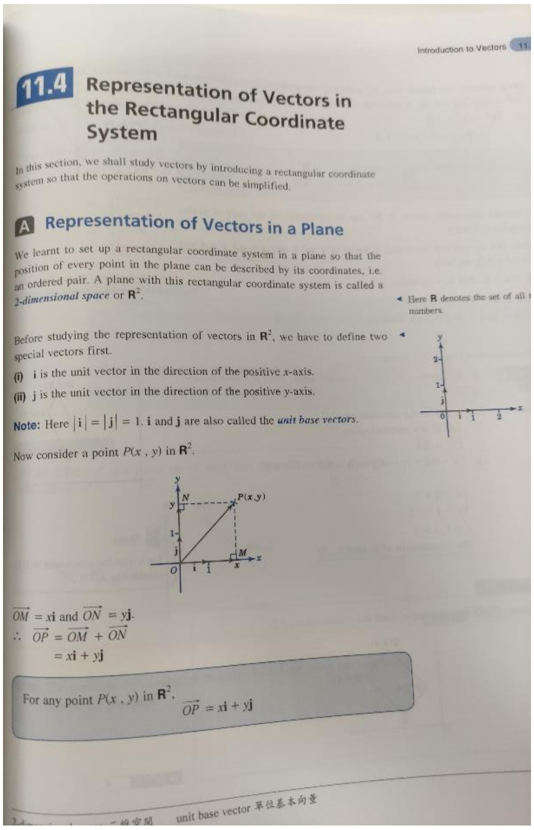
In three dimensions a displacement can be represented as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The diagram shows a three-dimensional set of axes, x, y, and z.

The column vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be represented as $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where **i**, **j**, and **k** are unit vectors in the x, y, and z directions respectively.

Vectors 183

UK Textbook example 1


Introduction to Vectors

11.4 Representation of Vectors in the Rectangular Coordinate System

In this section, we shall study vectors by introducing a rectangular coordinate system so that the operations on vectors can be simplified.

A Representation of Vectors in a Plane

We learnt to set up a rectangular coordinate system in a plane so that the position of every point in the plane can be described by its coordinates, i.e. an ordered pair. A plane with this rectangular coordinate system is called a *2-dimensional space* or $\mathbf{R}^2$.

◂ Here **R** denotes the set of all real numbers.

Before studying the representation of vectors in $\mathbf{R}^2$, we have to define two special vectors first.

(i) **i** is the unit vector in the direction of the positive x-axis.

(ii) **j** is the unit vector in the direction of the positive y-axis.

Note: Here $|\mathbf{i}| = |\mathbf{j}| = 1$. **i** and **j** are also called the ***unit base vectors***.

Now consider a point $P(x, y)$ in $\mathbf{R}^2$.

$\overrightarrow{OM} = x\mathbf{i}$ and $\overrightarrow{ON} = y\mathbf{j}$.

$\therefore \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{ON}$

$= x\mathbf{i} + y\mathbf{j}$

For any point $P(x, y)$ in $\mathbf{R}^2$, $\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$

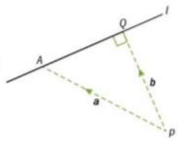
unit base vector 單位基本向量

HK Textbook example 1

2. UK textbooks teach students to find the distance from a point to a line in the form: $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ while Hong Kong textbooks teach vector projection of a vector onto another vector and do not cover the vector equation of line.

9.8 The distance from a point to a line

The distance from a point to a line is defined as the shortest distance, that is, the distance along a direction perpendicular to the line.



Let Q be the foot of the perpendicular from the point P to the line l, which has equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

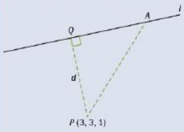
To find the distance from point P to line l, the coordinates of Q are found by using the fact that PQ is perpendicular to l. The direction vector from P to any point on the line can be found in terms of t, then the scalar product between this vector and the direction of l can be equated to 0 in order to find the value of t for point Q, the foot of the perpendicular from P to the line.

This value of t can then be substituted into $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ to find the position vector of Q and, hence, the distance PQ.

This process is demonstrated in Example 20.

Example 20

Find the distance of the point $P(3, 3, 1)$ from the line l whose equation is $\mathbf{r} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$.



Any point on the line, say Q, has position vector

$$\mathbf{q} = \begin{pmatrix} -4+t \\ 3+2t \\ -4t \end{pmatrix}.$$

Find the general position vector of a point on the line.

The direction of l is $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$

For Q to be the foot of the perpendicular from P to the line,

$$\overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 0$$

But $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{pmatrix} -4+t \\ 3+2t \\ -4t \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -7+t \\ 2t \\ -4t-1 \end{pmatrix}$

Find $\overrightarrow{PQ}$ in terms of t.

$$\begin{pmatrix} -7+t \\ 2t \\ -4t-1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 0$$

Substitute for $\overrightarrow{PQ}$ in $\overrightarrow{PQ} \cdot \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 0$.

$$(-7+t) \times 1 + (2t) \times 2 + (-4t-1) \times (-4) = 0$$
$$-7 + t + 4t + 16t + 4 = 0$$
$$t = \frac{1}{7}$$

Calculate the value of t.

So $\overrightarrow{PQ} = \begin{pmatrix} -7+\frac{1}{7} \\ 2\times\frac{1}{7} \\ -4\times\frac{1}{7}-1 \end{pmatrix} = \begin{pmatrix} -\frac{48}{7} \\ \frac{2}{7} \\ -\frac{11}{7} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -48 \\ 2 \\ -11 \end{pmatrix}$

Substitute for t in $\overrightarrow{PQ}$.

$$PQ = \frac{1}{7}\sqrt{(-48)^2 + 2^2 + (-11)^2} = \frac{\sqrt{2429}}{7}$$

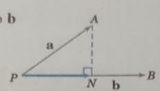
Calculate the length of PQ.

Note: If we want to find the position vector or coordinates of Q, we can substitute $t = \frac{1}{7}$ into $\mathbf{q} = \begin{pmatrix} -4+t \\ 3+2t \\ -4t \end{pmatrix}$.

UK Textbook example 2

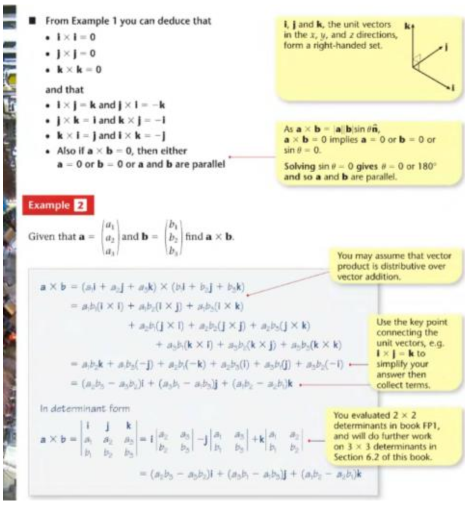
0 2B Chapter 12

	Note	Example
5.	**Vector Projection** Vector projection of **a** onto **b** $= \frac{\mathbf{a} \cdot \mathbf{b}}{\lvert\mathbf{b}\rvert^2}\mathbf{b}$ Projection of **a** onto **b** $= \frac{\lvert\mathbf{a} \cdot \mathbf{b}\rvert}{\lvert\mathbf{b}\rvert}$	Find the vector projection of $\mathbf{a} = \mathbf{i} - 2\mathbf{j}$ onto $\mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Vector projection of **a** onto **b** $= \frac{(1)(1) + (-2)(1) + (0)(-2)}{1^2 + 1^2 + (-2)^2}(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ $= -\frac{1}{6}\mathbf{i} - \frac{1}{6}\mathbf{j} + \frac{1}{3}\mathbf{k}$



HK Textbook example 2

3. Hong Kong textbooks put more emphasis on the properties of vector products and show more proofs of the properties while UK textbooks do not.


- From Example 1 you can deduce that
  - $\mathbf{i} \times \mathbf{i} = 0$
  - $\mathbf{j} \times \mathbf{j} = 0$
  - $\mathbf{k} \times \mathbf{k} = 0$

  and that
  - $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
  - $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
  - $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$
  - Also if $\mathbf{a} \times \mathbf{b} = 0$, then either $\mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel

i, j and k, the unit vectors in the x, y, and z directions, form a right-handed set.

As $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, $\mathbf{a} \times \mathbf{b} = 0$ implies $\mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\sin\theta = 0$.
Solving $\sin\theta = 0$ gives $\theta = 0$ or $180°$ and so $\mathbf{a}$ and $\mathbf{b}$ are parallel.

Example 2

Given that $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ find $\mathbf{a} \times \mathbf{b}$.

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1(\mathbf{i}\times\mathbf{i}) + a_1b_2(\mathbf{i}\times\mathbf{j}) + a_1b_3(\mathbf{i}\times\mathbf{k}) \\ &\quad + a_2b_1(\mathbf{j}\times\mathbf{i}) + a_2b_2(\mathbf{j}\times\mathbf{j}) + a_2b_3(\mathbf{j}\times\mathbf{k}) \\ &\quad + a_3b_1(\mathbf{k}\times\mathbf{i}) + a_3b_2(\mathbf{k}\times\mathbf{j}) + a_3b_3(\mathbf{k}\times\mathbf{k}) \\ &= a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3(\mathbf{i}) + a_3b_1(\mathbf{j}) + a_3b_2(-\mathbf{i}) \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

You may assume that vector product is distributive over vector addition.

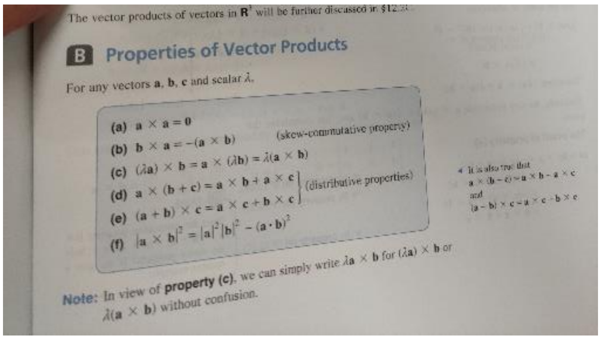
Use the key point connecting the unit vectors, e.g. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ to simplify your answer then collect terms.

In determinant form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

You evaluated 2 × 2 determinants in book FP1, and will do further work on 3 × 3 determinants in Section 6.2 of this book.

UK Textbook example 3


The vector products of vectors in $\mathbf{R}^3$ will be further discussed in §12.…

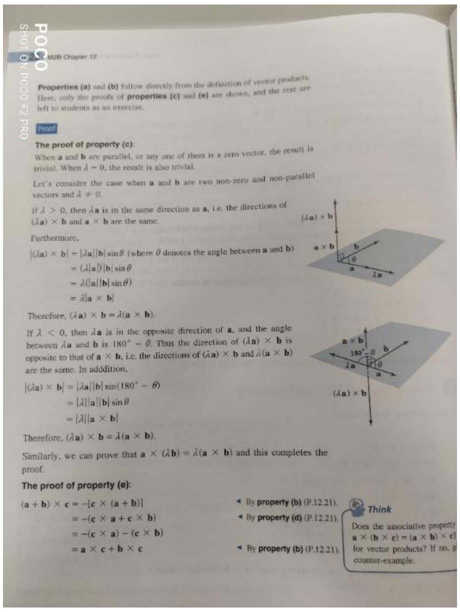
B Properties of Vector Products

For any vectors **a**, **b**, **c** and scalar λ,

(a) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
(b) $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$ (skew-commutative property)
(c) $(\lambda\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$
(d) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
(e) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ (distributive properties)
(f) $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$

It is also true that $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{c}$

Note: In view of **property (c)**, we can simply write $\lambda\mathbf{a} \times \mathbf{b}$ for $(\lambda\mathbf{a}) \times \mathbf{b}$ or $\lambda(\mathbf{a} \times \mathbf{b})$ without confusion.


Properties (a) and (b) follow directly from the definition of vector products. Here, only the proofs of properties (c) and (e) are shown, and the rest are left to students as an exercise.

Proof

The proof of property (c):

When **a** and **b** are parallel, or any one of them is a zero vector, the result is trivial. When $\lambda = 0$, the result is also trivial.

Let's consider the case when **a** and **b** are two non-zero and non-parallel vectors and $\lambda \neq 0$.

If $\lambda > 0$, then $\lambda\mathbf{a}$ is in the same direction as **a**, i.e. the directions of $(\lambda\mathbf{a}) \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are the same.

Furthermore,

$$\begin{aligned}|(\lambda\mathbf{a}) \times \mathbf{b}| &= |\lambda\mathbf{a}||\mathbf{b}|\sin\theta \text{ (where } \theta \text{ denotes the angle between } \mathbf{a} \text{ and } \mathbf{b}) \\ &= (\lambda|\mathbf{a}|)|\mathbf{b}|\sin\theta \\ &= \lambda(|\mathbf{a}||\mathbf{b}|\sin\theta) \\ &= \lambda|\mathbf{a} \times \mathbf{b}|\end{aligned}$$

Therefore, $(\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$.

If $\lambda < 0$, then $\lambda\mathbf{a}$ is in the opposite direction of **a**, and the angle between $\lambda\mathbf{a}$ and **b** is $180° - \theta$. Thus the direction of $(\lambda\mathbf{a}) \times \mathbf{b}$ is opposite to that of $\mathbf{a} \times \mathbf{b}$, i.e. the directions of $(\lambda\mathbf{a}) \times \mathbf{b}$ and $\lambda(\mathbf{a} \times \mathbf{b})$ are the same. In addition,

$$\begin{aligned}|(\lambda\mathbf{a}) \times \mathbf{b}| &= |\lambda\mathbf{a}||\mathbf{b}|\sin(180° - \theta) \\ &= |\lambda||\mathbf{a}||\mathbf{b}|\sin\theta \\ &= |\lambda||\mathbf{a} \times \mathbf{b}|\end{aligned}$$

Therefore, $(\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$.

Similarly, we can prove that $\mathbf{a} \times (\lambda\mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$ and this completes the proof.

The proof of property (e):

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= -[\mathbf{c} \times (\mathbf{a} + \mathbf{b})] && \text{By property (b) (P.12.21)} \\ &= -(\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}) && \text{By property (d) (P.12.21)} \\ &= -(\mathbf{c} \times \mathbf{a}) - (\mathbf{c} \times \mathbf{b}) \\ &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} && \text{By property (b) (P.12.21)}\end{aligned}$$

Think: Does the associative property $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ hold for vector products? If no, give a counter-example.

HK Textbook example 3

4. There are also many examples of UK textbooks contents that do not exist in Hong Kong M2 textbooks, such as the representation of a plane in scalar form, vector form or the Cartesian form (UK Textbook example 4), describing linear transformation by using matrices (UK Textbook example 5) and vector application to displacements, accelerations, forces (UK Textbook example 6).

5.5 You can write the equation of a plane in the scalar form r.n = p, or in the vector form r = a + sb + tc, or in the Cartesian form $ax + by + cz + d = 0$

Example 20

Given that the vector **n** is perpendicular to the plane Π and that Π passes through the point A with position vector **a**, find an equation of the plane Π.

Let point R be a point in the plane with position vector **r**,
then $\overrightarrow{AR}$ = **r** − **a**
As $\overrightarrow{AR}$ is a vector which lies in the plane, $\overrightarrow{AR}$.**n** = 0
So (**r** − **a**).**n** = 0
i.e. **r.n** = **a.n**
So if **a.n.** = p, where p is a scalar, then the equation of the plane Π is **r.n** = p.

The normal to the plane is perpendicular to all lines which lie in the plane, and so **n** is perpendicular to $\overrightarrow{AR}$.

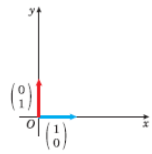
When two vectors are perpendicular their scalar product is zero.

■ **The scalar product form of the equation of a plane is r.n = a.n = p where n is normal to the plane, a is the position vector of a point in the plane and r is the position vector of a general point on the plane. p is a scalar constant.**

UK Textbook example 4

4.6 You can use matrices to represent rotations, reflections and enlargements.

- In GCSE you may have met some simple transformations such as rotations, reflections, enlargements and translations.
- A translation is *not* a linear transformation (since the origin moves) but all the others are and in this section we shall see how to represent them using matrices.
- To identify the matrix representing a particular transformation you should consider the effect of the matrix or the transformation on two simple vectors $\begin{pmatrix}1\\0\end{pmatrix}$ (sometimes denoted as **i**) and $\begin{pmatrix}0\\1\end{pmatrix}$ (sometimes denoted by **j**).



- Given any matrix $\mathbf{M} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ you can see that

$$\mathbf{M}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \quad \text{and} \quad \mathbf{M}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

so the first column of **M** gives the image of $\begin{pmatrix}1\\0\end{pmatrix}$ and the second column of **M** gives the image of $\begin{pmatrix}0\\1\end{pmatrix}$.

- You can use this information to identify the transformation represented by a matrix.

In FP1 you should be able to identify matrices representing the following linear transformations.

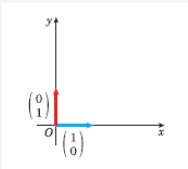
Rotation	about (0, 0) of angles that are multiples of 45°.
Enlargement	centre (0, 0) of scale factor $k (k \neq 0, k \in \mathbb{R})$.
Reflection	in coordinate axes or the lines $y = \pm x$.
Identity	the matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is called **I** and does not carry out any transformation. (This is equivalent to multiplying by 1 in arithmetic.)

Example 14

Describe fully the geometrical transformations represented by these matrices.

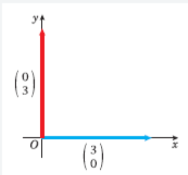
a $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$ **b** $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$ **c** $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

a



A diagram is very useful. Use it to show the images of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

Under the action of $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$, this becomes



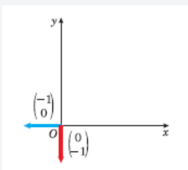
The images of $\begin{pmatrix}1\\0\end{pmatrix}$ (blue) and $\begin{pmatrix}0\\1\end{pmatrix}$ (red) are in the same directions as the original vectors but 3 times as long. This indicates an enlargement.

The transformation is therefore an enlargement, scale factor 3 and centre (0, 0).

When describing an enlargement you should state the scale factor and the centre (always (0, 0)).

b

Draw a diagram showing the images of $\begin{pmatrix}1\\0\end{pmatrix}$ (blue) and $\begin{pmatrix}0\\1\end{pmatrix}$ (red).



The vector $\begin{pmatrix}1\\0\end{pmatrix}$ has moved to $\begin{pmatrix}-1\\0\end{pmatrix}$. This could be due to a reflection in $x = 0$ or a rotation of 180°.

The vector $\begin{pmatrix}0\\1\end{pmatrix}$ has moved to $\begin{pmatrix}0\\-1\end{pmatrix}$. This could be due to a reflection in $y = 0$ or a rotation of 180°. Since the same transformation has caused both movements it must be a rotation.

This transformation is therefore a rotation of 180° (anticlockwise) about (0, 0).

N.B. This transformation could also be described as an enlargement centre (0, 0) scale factor −1.

87

UK Textbook example 5

6.5 You can express the velocity of a particle as a vector.

- **The velocity of a particle is a vector in the direction of motion. Its magnitude is the speed of the particle. The velocity is usually denoted by v.**

If a particle is moving with constant velocity $\mathbf{v}$ m s^{-1}, then after time t seconds it will have moved $\mathbf{v}t$ m. The displacement is parallel to the velocity. The magnitude of the displacement is the distance from the starting point.

6.6 You can solve problems involving velocity and time using vectors.

- **If a particle starts from the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, then its displacement from its initial position at time t is $\mathbf{v}t$ and its position vector $\mathbf{r}$ is given by**

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$$

Example 12

A particle starts from the point with position vector $(3\mathbf{i} + 7\mathbf{j})$ m and moves with constant velocity $(2\mathbf{i} - \mathbf{j})$ m s^{-1}. Find the position vector of the particle 4 seconds later.

Displacement $= \mathbf{v}t = 4(2\mathbf{i} - \mathbf{j}) = 8\mathbf{i} - 4\mathbf{j}$ — Displacement $= \mathbf{v} \times t$

Position vector $\mathbf{r} = (3\mathbf{i} + 7\mathbf{j}) + (8\mathbf{i} - 4\mathbf{j})$ — $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$

$= (3 + 8)\mathbf{i} + (7 - 4)\mathbf{j}$

$= 11\mathbf{i} + 3\mathbf{j}$ — Position vector after 4 seconds = position vector of starting point + displacement

6.7 You can use vectors to solve problems about forces.

- **If a particle is resting in equilibrium then the resultant of all the forces acting on it is zero. This means the sum of the vectors of the forces is the zero vector.**

Example 17

The forces $2\mathbf{i} + 3\mathbf{j}$, $4\mathbf{i} - \mathbf{j}$, $-3\mathbf{i} + 2\mathbf{j}$ and $a\mathbf{i} + b\mathbf{j}$ act on a particle which is in equilibrium. Find the values of a and b.

$(2\mathbf{i} + 3\mathbf{j}) + (4\mathbf{i} - \mathbf{j}) + (-3\mathbf{i} + 2\mathbf{j}) + (a\mathbf{i} + b\mathbf{j}) = 0$ — If the particle is in equilibrium then the resultant force will be zero.

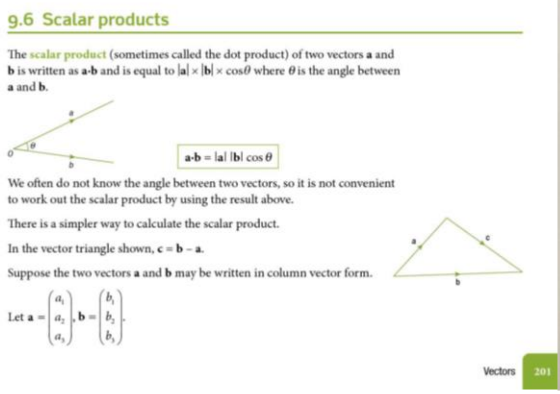
$(2 + 4 - 3 + a)\mathbf{i} + (3 - 1 + 2 + b)\mathbf{j} = 0$

$\Rightarrow \quad 3 + a = 0$ and $4 + b = 0$

$\Rightarrow \quad a = -3$ and $b = -4$

UK Textbook example 6

Some common examples in the textbooks of Hong Kong M2 and UK qualifications, such as the scalar products (UK & HK Textbook example 1), vector product (UK & HK Textbook example 2), interpret area of triangle and parallelogram with vectors (UK & HK Textbook example 3) and triple scalar products (UK & HK Textbook example 4).



9.6 Scalar products

The scalar product (sometimes called the dot product) of two vectors **a** and **b** is written as **a**·**b** and is equal to |**a**| × |**b**| × cos θ where θ is the angle between **a** and **b**.

a·**b** = |**a**| |**b**| cos θ

We often do not know the angle between two vectors, so it is not convenient to work out the scalar product by using the result above.

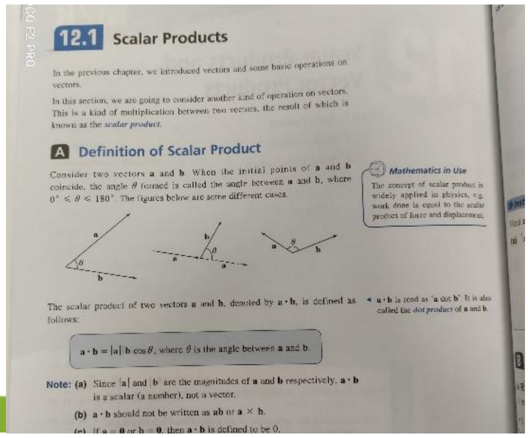
There is a simpler way to calculate the scalar product.

In the vector triangle shown, **c** = **b** − **a**.

Suppose the two vectors **a** and **b** may be written in column vector form.

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Vectors 201



12.1 Scalar Products

In the previous chapter, we introduced vectors and some basic operations on vectors.

In this section, we are going to consider another kind of operation on vectors. This is a kind of multiplication between two vectors, the result of which is known as the *scalar product*.

A Definition of Scalar Product

Consider two vectors **a** and **b**. When the initial points of **a** and **b** coincide, the angle θ formed is called the angle between **a** and **b**, where $0° \le \theta \le 180°$. The figures below are some different cases.

Mathematics in Use: The concept of scalar product is widely applied in physics, e.g. work done is equal to the scalar product of force and displacement.

The scalar product of two vectors **a** and **b**, denoted by **a** · **b**, is defined as follows:

a · **b** = |**a**| |**b**| cos θ, where θ is the angle between **a** and **b**.

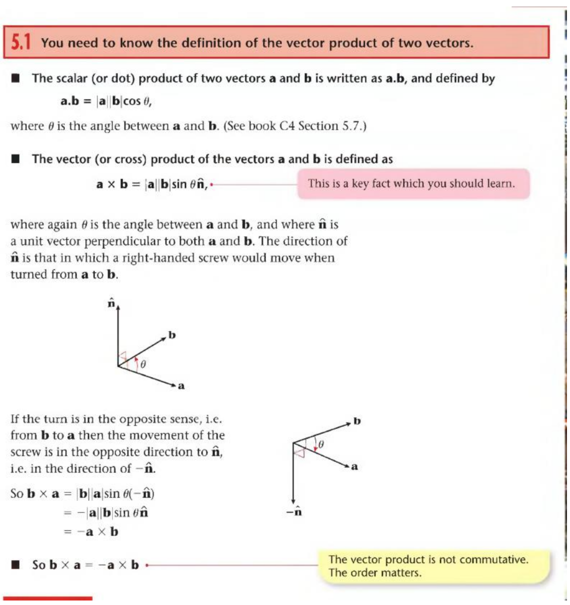
a · **b** is read as 'a dot b'. It is also called the *dot product* of **a** and **b**.

Note: (a) Since |**a**| and |**b**| are the magnitudes of **a** and **b** respectively, **a** · **b** is a scalar (a number), not a vector.

(b) **a** · **b** should not be written as **ab** or **a** × **b**.

(c) If **a** = **0** or **b** = **0**, then **a** · **b** is defined to be 0.

UK & HK Textbook example 1



5.1 You need to know the definition of the vector product of two vectors.

- The scalar (or dot) product of two vectors **a** and **b** is written as **a.b**, and defined by

  a.b = |**a**||**b**|cos θ,

  where θ is the angle between **a** and **b**. (See book C4 Section 5.7.)

- The vector (or cross) product of the vectors **a** and **b** is defined as

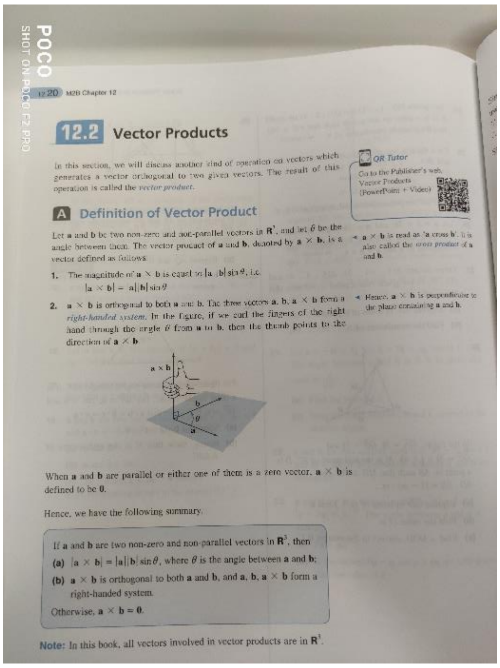
  a × **b** = |**a**||**b**|sin θ **n̂**, ← This is a key fact which you should learn.

  where again θ is the angle between **a** and **b**, and where **n̂** is a unit vector perpendicular to both **a** and **b**. The direction of **n̂** is that in which a right-handed screw would move when turned from **a** to **b**.

If the turn is in the opposite sense, i.e. from **b** to **a** then the movement of the screw is in the opposite direction to **n̂**, i.e. in the direction of −**n̂**.

So **b** × **a** = |**b**||**a**|sin θ(−**n̂**)
= −|**a**||**b**|sin θ**n̂**
= −**a** × **b**

- So **b** × **a** = −**a** × **b** ← The vector product is not commutative. The order matters.



12 20 M2B Chapter 12

12.2 Vector Products

In this section, we will discuss another kind of operation on vectors which generates a vector orthogonal to two given vectors. The result of this operation is called the *vector product*.

QR Tutor: Go to the Publisher's web. Vector Products (PowerPoint + Video)

A Definition of Vector Product

Let **a** and **b** be two non-zero and non-parallel vectors in $\mathbf{R}^3$, and let θ be the angle between them. The vector product of **a** and **b**, denoted by **a** × **b**, is a vector defined as follows.

a × **b** is read as 'a cross b'. It is also called the *cross product* of **a** and **b**.

1. The magnitude of **a** × **b** is equal to |**a**||**b**| sin θ, i.e.
   |**a** × **b**| = |**a**||**b**| sin θ
2. **a** × **b** is orthogonal to both **a** and **b**. The three vectors **a**, **b**, **a** × **b** form a *right-handed system*. In the figure, if we curl the fingers of the right hand through the angle θ from **a** to **b**, then the thumb points to the direction of **a** × **b**.

Hence, **a** × **b** is perpendicular to the plane containing **a** and **b**.

When **a** and **b** are parallel or either one of them is a zero vector, **a** × **b** is defined to be **0**.

Hence, we have the following summary.

If **a** and **b** are two non-zero and non-parallel vectors in $\mathbf{R}^3$, then
(a) |**a** × **b**| = |**a**||**b**| sin θ, where θ is the angle between **a** and **b**;
(b) **a** × **b** is orthogonal to both **a** and **b**, and **a**, **b**, **a** × **b** form a right-handed system.
Otherwise, **a** × **b** = **0**.

Note: In this book, all vectors involved in vector products are in $\mathbf{R}^3$.

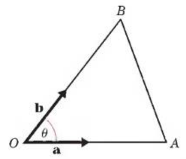
UK & HK Textbook example 2

5.2 You can interpret $|\mathbf{a} \times \mathbf{b}|$ as an area.

The vector product has a number of applications. For example, if you study the M5 specification you will use it to find moments of forces. Later in this chapter you will use it to give an alternative form of the equation of a straight line in vector form and to find the shortest distance between skew lines (i.e. lines which do not meet and are not parallel).

In this section you will find areas of triangles and parallelograms using vector products.

Example 6



Find the area of triangle OAB, where O is the origin, A is the point with position vector $\mathbf{a}$ and B is the point with position vector $\mathbf{b}$.

$$\text{Area of triangle } OAB = \tfrac{1}{2} OA.OB \sin\theta$$
$$= \tfrac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta$$
$$= \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}|$$

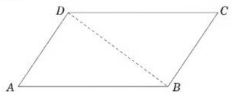
Use the trigonometric formula area of triangle $= \frac{1}{2}ab\sin C$, and let the angle $AOB = \theta$.

You use the definition of vector product to obtain this result.

■ **Area of triangle** $OAB = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

107

Example 8



Find the area of the parallelogram $ABCD$, where the position vectors of A, B and D are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ respectively.

area of parallelogram $ABCD$
$= $ area of triangle ABD + area of triangle BCD
$= 2 \times$ area of triangle ABD
$= AB \cdot AD \sin\theta$
$= |\mathbf{b} - \mathbf{a}||\mathbf{d} - \mathbf{a}|\sin\theta$
$= |(\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})|$
$= |(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{d}) + (\mathbf{d} \times \mathbf{a})|$

As the two triangles are congruent and have equal area.

θ is the angle $B\hat{A}D$.

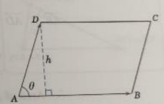
■ **Area of parallelogram** $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$
$= |(\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})|$
$= |(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{d}) + (\mathbf{d} \times \mathbf{a})|$

108

II. Areas of Parallelograms and Triangles

Vector products can also be applied to find the areas of parallelograms and triangles.

Consider the parallelogram $ABCD$, where AB is the base and h is the height.



Area of $ABCD = AB(h)$
$= AB(AD\sin\theta)$
$= |\overrightarrow{AB}||\overrightarrow{AD}|\sin\theta$
$= |\overrightarrow{AB} \times \overrightarrow{AD}|$ ◂ By the definition of vector product.

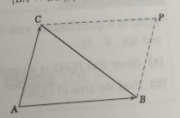
Hence, we have

Area of parallelogram $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

◂ The area is also equal to $|\overrightarrow{BA} \times \overrightarrow{BC}|$, $|\overrightarrow{CB} \times \overrightarrow{CD}|$, etc.

Now, let's turn to the area of triangle.

For any $\triangle ABC$, construct the parallelogram $ABPC$ as shown in the figure.



Then

area of $\triangle ABC = \frac{1}{2}$(area of parallelogram $ABPC$).

Hence, we have

Area of $\triangle ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

◂ The area is also equal to $\frac{1}{2}|\overrightarrow{BA} \times \overrightarrow{BC}|$ and $\frac{1}{2}|\overrightarrow{CA} \times \overrightarrow{CB}|$.

Note: When the area of $\triangle ABC$ is 0 (i.e. $\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{0}$), it is obvious that A, B and C are collinear (i.e. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel). In fact, for non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then $\mathbf{a}$ and $\mathbf{b}$ are parallel.

UK & HK Textbook example 3

5.3 You can find the triple scalar product **a**.(**b** × **c**) of three vectors **a**, **b** and **c**, and use it to find the volume of a parallelepiped and of a tetrahedron.

You know that $\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\mathbf{i} + (b_3c_1 - b_1c_3)\mathbf{j} + (b_1c_2 - b_2c_1)\mathbf{k}$,
where $\mathbf{b} = (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$ and $c = (c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})$.

So if $\mathbf{a} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$ then

■ $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

This can also be written as

■ $$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

a.(**b** × **c**) is known as a **triple scalar product**.

12.3 Scalar Triple Products

For any three vectors **a**, **b** and **c** in $\mathbf{R}^3$, the product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the ***scalar triple product*** of **a**, **b** and **c**. The result of this product is a scalar.

Let $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ and $\mathbf{c} = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$.

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}\mathbf{k}$$

$$\therefore\ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = x_1\begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - y_1\begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Thus, for vectors in $\mathbf{R}^3$, scalar triple products can be computed directly by the following formula.

If $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ and $\mathbf{c} = x_3\mathbf{i} + y_3\mathbf{j} + z_3\mathbf{k}$, then

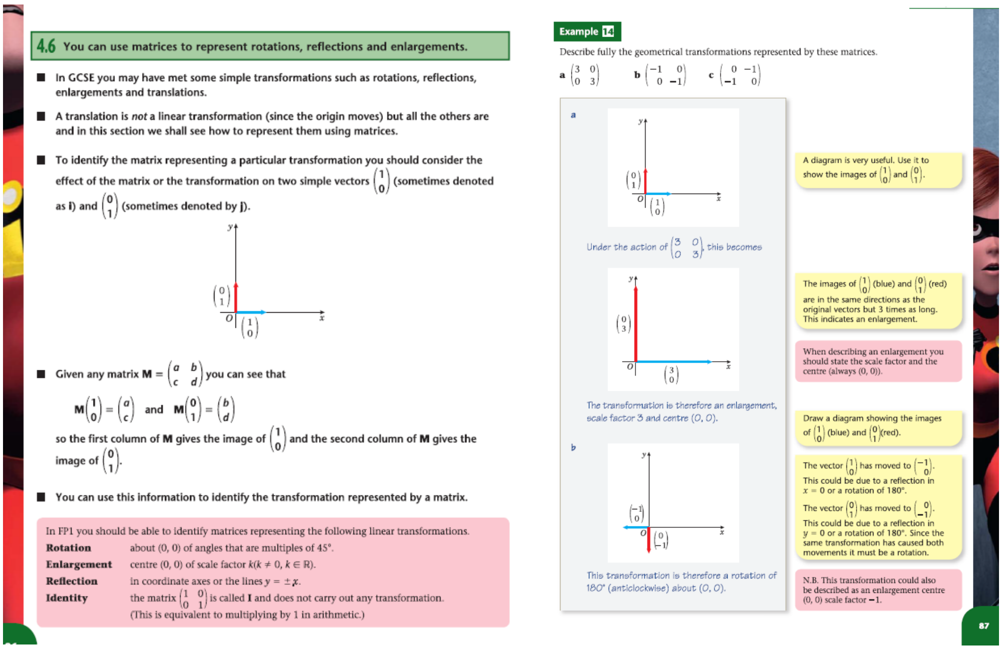
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

UK & HK Textbook example 4

Content in linear algebra covers around 11% in CIE A level Pure Mathematics, 62% in Edexcel IALs Further Mathematics, 10% in Edexcel IALs Mechanic and 55% in HKDSE M2 textbooks. The percentage of linear algebra content in UK qualifications is more than that in HKDSE M2 and therefore the UK qualifications put more emphasis on linear algebra. Furthermore, the teaching order of linear algebra teaching contents in UK qualifications mathematics subject and HKDSE M2 is different. For CIE A level or Edexcel IALs mathematics or further mathematics, vectors would be taught before teaching matrices but vectors would be taught after teaching matrices in the syllabus of HKDSE M2.

Although the syllabus of HKDSE M2 recommends teachers put more emphasis on the geometric representation relating to linear algebra, the number of diagrams is not enough to explain the contents and the textbooks do not cover the geometric representation of linear transformation.

For the characteristic of the textbooks, UK textbooks show more geometric representation in matrix or vectors (Figure 22). They also require students to draw the diagrams to show the linear transformation under the matrix multiplication so that they are more interested in visualizing the abstract concept in linear algebra (Hege & Polthier, 2013). Only calculating the matrix multiplication is just enough for students to apply the skills but not achieve understanding or any higher level of learning objectives listed in Bloom's taxonomy (Armstrong, 2016).



4.6 You can use matrices to represent rotations, reflections and enlargements.

- In GCSE you may have met some simple transformations such as rotations, reflections, enlargements and translations.
- A translation is *not* a linear transformation (since the origin moves) but all the others are and in this section we shall see how to represent them using matrices.
- To identify the matrix representing a particular transformation you should consider the effect of the matrix or the transformation on two simple vectors $\begin{pmatrix}1\\0\end{pmatrix}$ (sometimes denoted as **i**) and $\begin{pmatrix}0\\1\end{pmatrix}$ (sometimes denoted by **j**).
- Given any matrix $\mathbf{M} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$ you can see that

$$\mathbf{M}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix} \quad \text{and} \quad \mathbf{M}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

so the first column of **M** gives the image of $\begin{pmatrix}1\\0\end{pmatrix}$ and the second column of **M** gives the image of $\begin{pmatrix}0\\1\end{pmatrix}$.

- You can use this information to identify the transformation represented by a matrix.

In FP1 you should be able to identify matrices representing the following linear transformations.

Rotation	about (0, 0) of angles that are multiples of 45°.
Enlargement	centre (0, 0) of scale factor $k (k \neq 0, k \in \mathbb{R})$.
Reflection	in coordinate axes or the lines $y = \pm x$.
Identity	the matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is called **I** and does not carry out any transformation. (This is equivalent to multiplying by 1 in arithmetic.)

Example 14

Describe fully the geometrical transformations represented by these matrices.

a $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$ **b** $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$ **c** $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

a Under the action of $\begin{pmatrix}3 & 0\\0 & 3\end{pmatrix}$, this becomes

The transformation is therefore an enlargement, scale factor 3 and centre (0, 0).

b This transformation is therefore a rotation of 180° (anticlockwise) about (0, 0).

A diagram is very useful. Use it to show the images of $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

The images of $\begin{pmatrix}1\\0\end{pmatrix}$ (blue) and $\begin{pmatrix}0\\1\end{pmatrix}$ (red) are in the same directions as the original vectors but 3 times as long. This indicates an enlargement.

When describing an enlargement you should state the scale factor and the centre (always (0, 0)).

Draw a diagram showing the images of $\begin{pmatrix}1\\0\end{pmatrix}$ (blue) and $\begin{pmatrix}0\\1\end{pmatrix}$ (red).

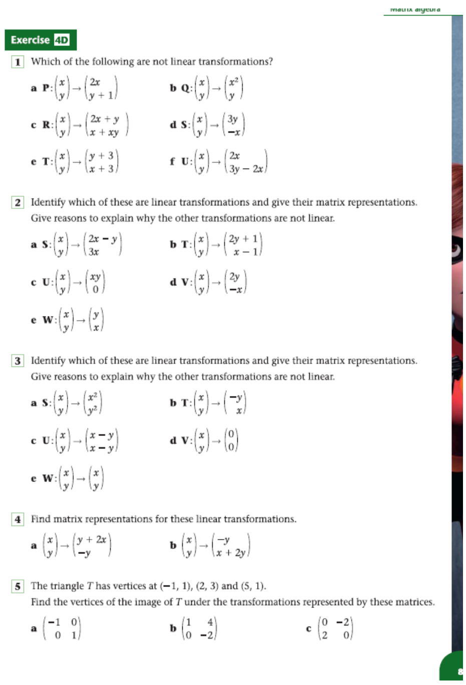
The vector $\begin{pmatrix}1\\0\end{pmatrix}$ has moved to $\begin{pmatrix}-1\\0\end{pmatrix}$. This could be due to a reflection in $x = 0$ or a rotation of 180°.
The vector $\begin{pmatrix}0\\1\end{pmatrix}$ has moved to $\begin{pmatrix}0\\-1\end{pmatrix}$. This could be due to a reflection in $y = 0$ or a rotation of 180°. Since the same transformation has caused both movements it must be a rotation.

N.B. This transformation could also be described as an enlargement centre (0, 0) scale factor −1.

87

Figure 22. Example of geometric representation in vectors in UK textbooks

From the observation, the percentage of drilling in UK textbooks is fewer than that of Hong Kong textbooks. There are usually 7-10 exercise questions in UK textbooks and 15-20 questions in Hong Kong textbooks (Figure 23). Fewer drilling questions can reduce the burden of students but the exercise questions in Hong Kong textbooks are separated into 2 levels to take care of learning differences. Students have more flexibility to choose their own exercises base on their abilities. Moreover, students in the UK are encouraged to think and try before they learn since very few instructions are written beside the examples and questions posed in the opening of the chapter (Figure 24). Some students may find that it is like a riddle and be interested in learning the new chapter.



Matrix algebra

Exercise 4D

1 Which of the following are not linear transformations?

a $\mathbf{P}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}2x\\y+1\end{pmatrix}$ b $\mathbf{Q}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x^2\\y\end{pmatrix}$

c $\mathbf{R}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}2x+y\\x+xy\end{pmatrix}$ d $\mathbf{S}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}3y\\-x\end{pmatrix}$

e $\mathbf{T}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}y+3\\x+3\end{pmatrix}$ f $\mathbf{U}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}2x\\3y-2x\end{pmatrix}$

2 Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a $\mathbf{S}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}2x-y\\3x\end{pmatrix}$ b $\mathbf{T}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}2y+1\\x-1\end{pmatrix}$

c $\mathbf{U}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}xy\\0\end{pmatrix}$ d $\mathbf{V}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}2y\\-x\end{pmatrix}$

e $\mathbf{W}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}y\\x\end{pmatrix}$

3 Identify which of these are linear transformations and give their matrix representations. Give reasons to explain why the other transformations are not linear.

a $\mathbf{S}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x^2\\y^2\end{pmatrix}$ b $\mathbf{T}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}-y\\x\end{pmatrix}$

c $\mathbf{U}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x-y\\x-y\end{pmatrix}$ d $\mathbf{V}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}0\\0\end{pmatrix}$

e $\mathbf{W}:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\y\end{pmatrix}$

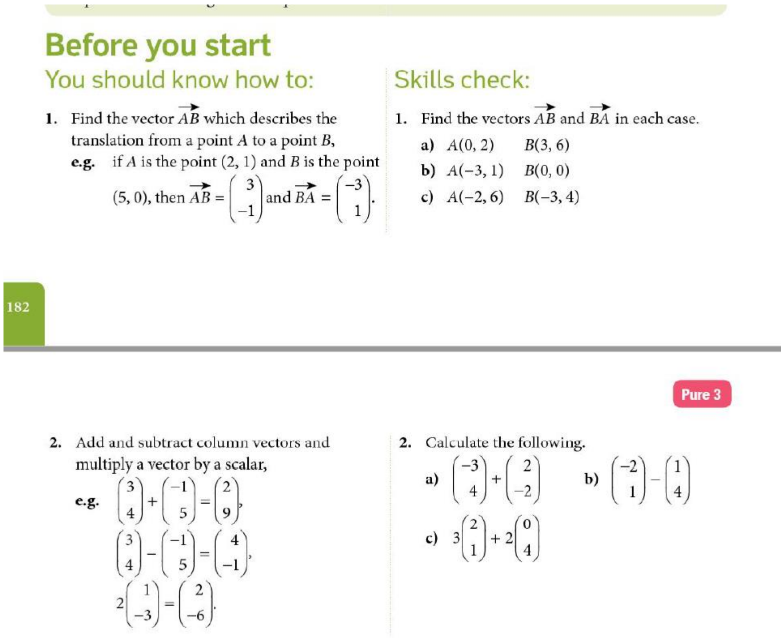
4 Find matrix representations for these linear transformations.

a $\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}y+2x\\-y\end{pmatrix}$ b $\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}-y\\x+2y\end{pmatrix}$

5 The triangle T has vertices at $(-1, 1)$, $(2, 3)$ and $(5, 1)$.
Find the vertices of the image of T under the transformations represented by these matrices.

a $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ b $\begin{pmatrix}1&4\\0&-2\end{pmatrix}$ c $\begin{pmatrix}0&-2\\2&0\end{pmatrix}$

Figure 23. Example of exercise questions in UK textbooks



Before you start

You should know how to:

1. Find the vector $\overrightarrow{AB}$ which describes the translation from a point A to a point B,
e.g. if A is the point $(2, 1)$ and B is the point $(5, 0)$, then $\overrightarrow{AB}=\begin{pmatrix}3\\-1\end{pmatrix}$ and $\overrightarrow{BA}=\begin{pmatrix}-3\\1\end{pmatrix}$.

2. Add and subtract column vectors and multiply a vector by a scalar,
e.g. $\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}-1\\5\end{pmatrix}=\begin{pmatrix}2\\9\end{pmatrix}$,
$\begin{pmatrix}3\\4\end{pmatrix}-\begin{pmatrix}-1\\5\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}$,
$2\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}2\\-6\end{pmatrix}$.

Skills check:

1. Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ in each case.
a) $A(0, 2)$ $B(3, 6)$
b) $A(-3, 1)$ $B(0, 0)$
c) $A(-2, 6)$ $B(-3, 4)$

2. Calculate the following.
a) $\begin{pmatrix}-3\\4\end{pmatrix}+\begin{pmatrix}2\\-2\end{pmatrix}$ b) $\begin{pmatrix}-2\\1\end{pmatrix}-\begin{pmatrix}1\\4\end{pmatrix}$
c) $3\begin{pmatrix}2\\1\end{pmatrix}+2\begin{pmatrix}0\\4\end{pmatrix}$

182

Pure 3

Figure 24. Example of opening questions in UK textbooks

On the other hand, the Hong Kong textbooks implement some class activities to allow students to apply their knowledge and explore the contents (Figure 25). It is beneficial for students to work or apply the skills while they learn and the teachers can immediately test their understanding of the topics. Also, Hong Kong textbooks show more proofs of theories than UK textbooks so that they can found out the origin of the theories. The logical reasoning and deduction of students can be enhanced and they can acquire the skill of organizing arguments to make mathematical conclusions (Reid & Knipping, 2010).

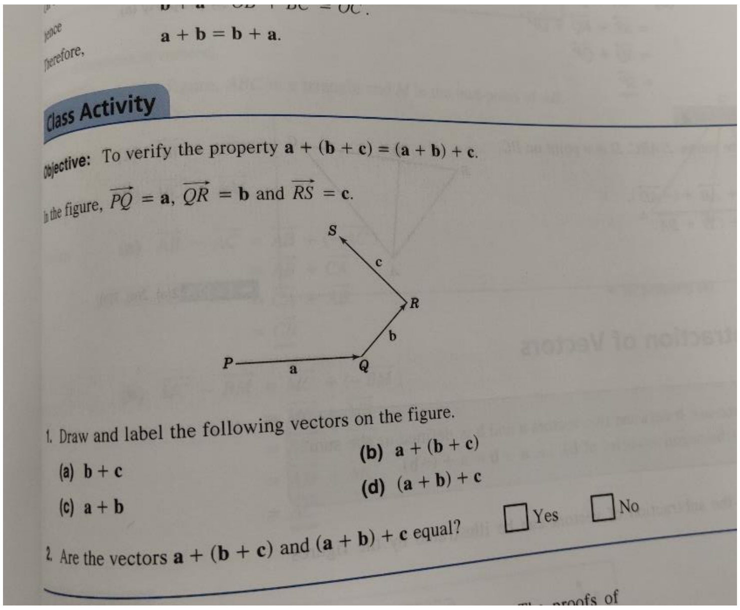


Figure 25. Example of class activity in HK textbooks

Through the comparison with the textbooks in the UK, Hong Kong textbooks can only show the formulas, methods of calculation and examples of problem-solving but they cannot explain the mathematical concept in linear algebra or visualize the concepts of vectors, matrices or linear transformation in geometric representations (Dogan-Dunlap, 2010). They put more emphasis on the drilling part. Memorizing and applying different formulas to calculate or solving problems is not enough for students to fully understand the fields of linear algebra.

Data analysis

The data of this research is collected from questionnaires and interviews. For the interview, a total of 6 participants joined the interview via Zoom and 40 questionnaires are collected. The consent form, questionnaire, interview scope and transcripts of interviews are listed in Appendix 1-4.

3 students who joined the UK qualifications (UK students) and 3 students who joined HKDSE M2 (HK students) participated in the interview for this research. Their mathematical ability and knowledge in linear algebra are different. They may have different comments and observations towards the difference of linear algebra contents between UK qualifications and HKDSE M2. Most of the participants prefer reading UK textbooks because UK textbooks cover more geometric representations, more inspirational questions and real

examples, fewer proofs on theories and drillings than HK textbooks. Some HK students may prefer using HK textbooks because they think the proofs in textbooks can enhance their mathematical reasoning and skills of proving by induction. For the recommendation towards Hong Kong secondary mathematics education, they suggest considering HKDSE M2 as an elective in the secondary curriculum, providing more learning hours and lesson time of HKDSE M2 and implementing more vector or matrix application examples to optimize the learning objectives and outcomes of secondary mathematics education. Although half of them replied that linear algebra is useful or beneficial in future study or careers, all of them agree that the awareness of linear algebra should be raised.

The data collected from 40 questionnaires completed by participants who took HKDSE M2 before are shown in the following diagrams:

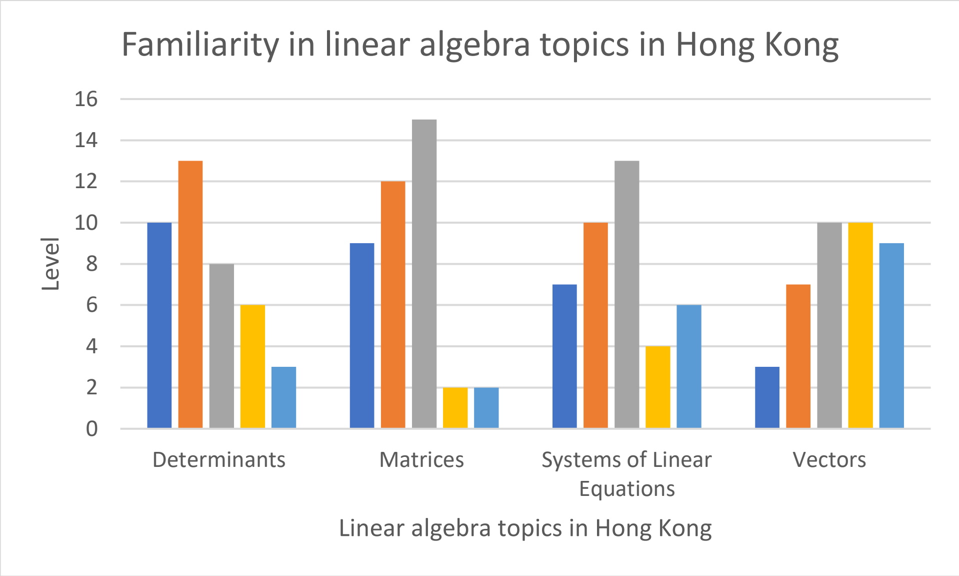


Diagram 1. Data analysis on participants' familiarity in linear algebra topics in Hong Kong

Diagram 1 indicates that the familiarity of participants keeps declining with the teaching order of linear algebra topics in Hong Kong. Participants possess less confidence in later teaching topics because of the level of difficulty, previous knowledge required or the lesson time of the topics.

The top 4 topics extracted from the UK textbook and have learned by the participants:	
displacement vectors $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} or\ 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$	singular matrix/ non-singular matrix If $det(\mathbf{A}) = 0$, then $\mathbf{A}$ is a singular matrix and $\mathbf{A}^{-1}$ cannot be found. If $det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is a non-singular matrix and $\mathbf{A}^{-1}$ exists.
general Cartesian equation of a straight line is	An eigenvector of a matrix $\mathbf{A}$ is a non-zero column vector $\mathbf{x}$ which

$\frac{(x-x_1)}{l}=\frac{(y-y_1)}{m}=\frac{(z-z_1)}{n}=\lambda,$ where the lines passes through the point (x_1,y_1,z_1), has direction ratios $l:m:n,$ and where λ is a parameter.	satisfies the equation $\mathbf{Ax}=\lambda\mathbf{x},$ where λ is a scalar which is the corresponding eigenvalue of the matrix.

The above table shows more than 20% of participants learned the above contents in linear algebra from the courses in universities. Very few participants learned them through websites or self-study. Many participants may not have learned the above contents not covered in the HKDSE M2 syllabus.

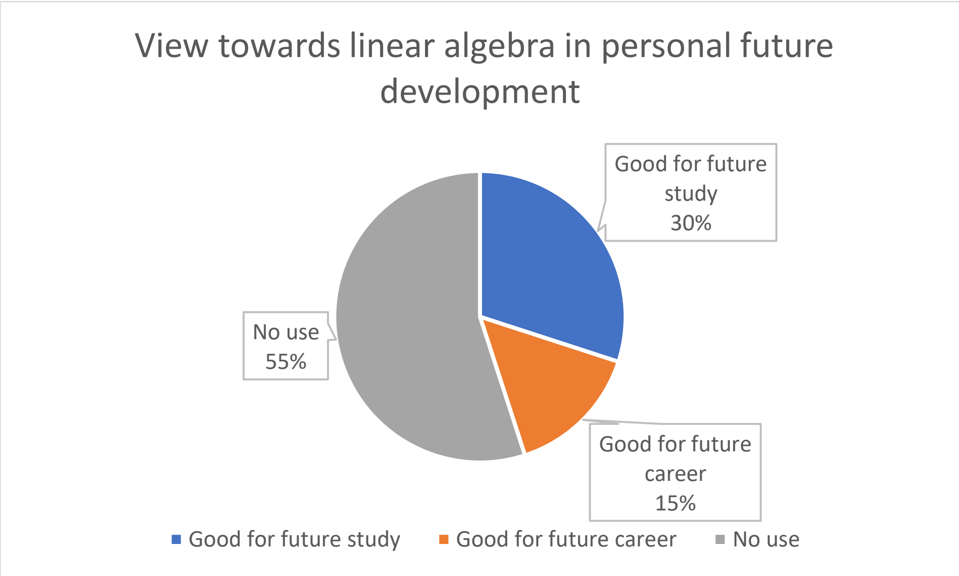


Diagram 2. Data analysis on view towards linear algebra in personal future development

Diagram 2 shows more than half of the participants think that linear algebra has no usage in their future, 30% of them think learning linear algebra is good for future study and 15% think linear algebra is good for future careers.

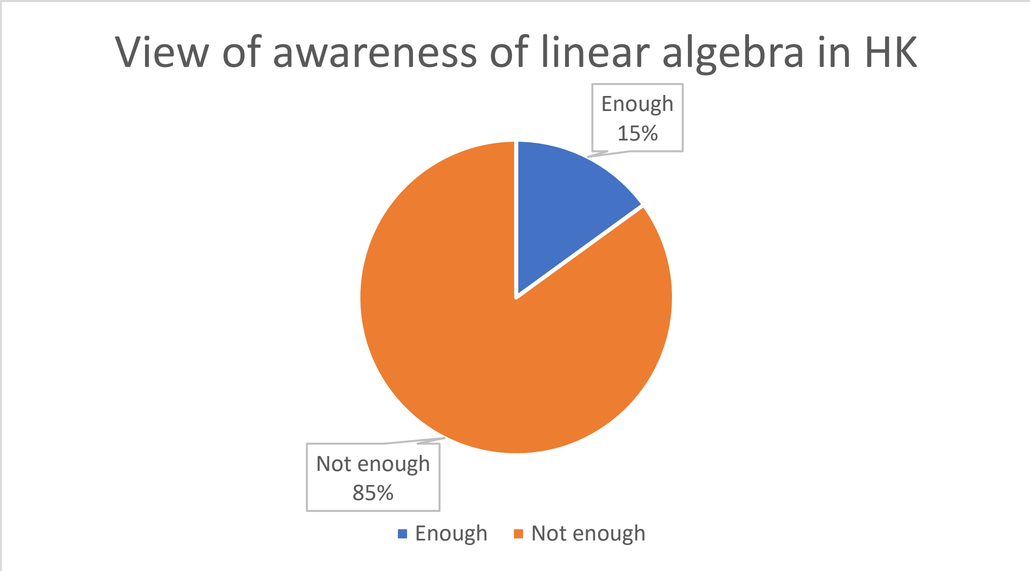


Diagram 3. Data analysis on view of awareness of linear algebra

Diagram 3 shows 85% of the participants think the awareness of linear algebra is not enough in Hong Kong mathematics education.

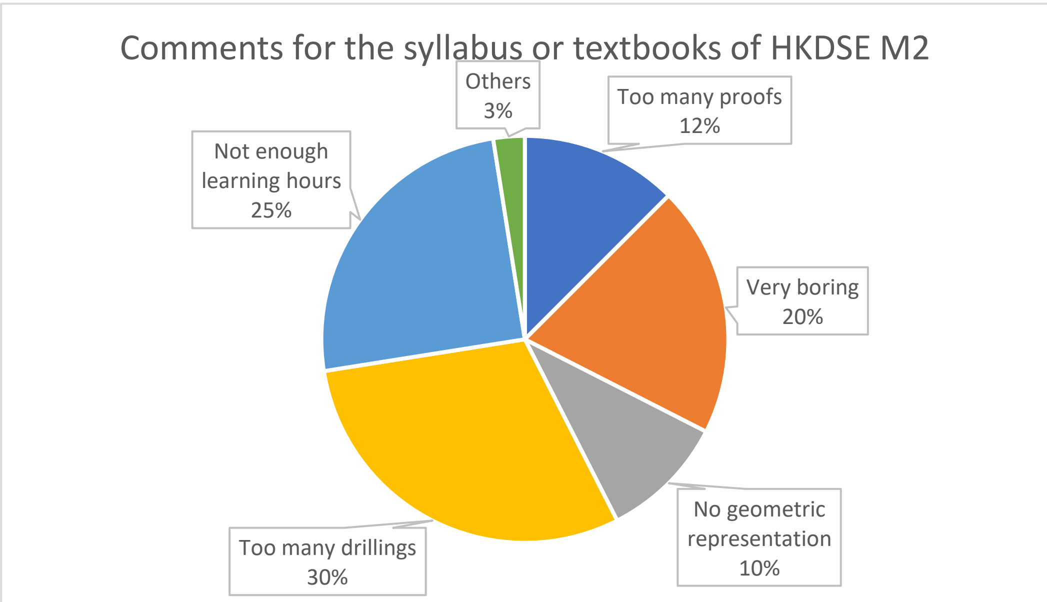


Diagram 4. Comments for the syllabus or textbooks of HKDSE M2

Diagram 4 indicates the majority of participants have negative views towards the syllabus or textbooks of HKDSE M2. They reply the HKDSE M2 textbook is very boring, covers too many proofs and drillings but no geometric representations. Some of them even complain the number of learning hours of HKDSE M2 is not enough.

Recommendation

With the collaboration of the comparison between the syllabus and textbook contents in linear algebra of HKDSE M2 and UK qualifications math papers, the interview and questionnaires result, some recommendation could be suggested for the syllabus, textbook contents of HKDSE M2, teachers and the curriculum of M2.

For the syllabus, solving the system of linear equations is unique in HKDSE M2 textbooks and it is an important topic in linear algebra. The Gaussian elimination, Cramer's rule and the inverse matrices method are three fundamental methods of matrix operation and solving linear equations or matrices of any size. However, the matrix transformation is not covered and should be added to the syllabus of the HKDSE M2. The HK students may not have a clear imagination of matrix operation in the 2D or 3D space. The matrices cannot be visualized and therefore the learning objectives and outcome may be degraded (Arcavi, 2003). The teaching order of vectors and matrices in the UK and Hong Kong are different but both have advantages.

Since UK textbooks denote the displacement vectors in both algebra and matrix form, students can learn the matrix operation and vector operation at the same time. For example, the addition and subtraction of matrices and vectors. But Hong Kong textbooks only denote vectors in algebra form. The knowledge of matrix cannot be applied in the following topics vectors, except the triple scalar product.

Apart from the suggestions for the syllabus, HK textbooks should also make some modifications to fit into the learning needs of local students. According to the comments from the local students, HK textbooks should reduce the number of proofs on theories or put a part of proofs in the supplementary materials of textbooks. HK textbooks should try to put in the examples with no or few instructions in the opening so that the implicit learning and learning space allows students to explore by themselves, which is more beneficial than spoon-feeding in books (Kolb & Kolb, 2005). The drilling questions in the textbooks can take care learning differences of local students. Teachers should provide more guidance for the students in drilling to further enhance their problem-solving skills.

Regarding the curriculum, the awareness of linear algebra in the UK have been grown higher than that in Hong Kong. More students in the UK are required to learn vectors than Hong Kong since it is included in one of the compulsory math papers in the UK. Hong Kong can learn from the UK math curriculum to implement some linear algebra topics in the compulsory syllabus to raise the awareness of linear algebra. Not only the compulsory part, but the vectors can also be implemented in the science subject. The Mechanic syllabus in UK qualifications demonstrates an example to apply vectors in displacements, accelerations, forces. More topics in linear algebra can be implemented in STEAM education, Physics or Mechanics in Science education so that the awareness of linear algebra can be raised and students can notice the importance of linear algebra.

In the long run, the extended part of mathematics (HKDSE M1 or M2) should be considered to develop as a regular elective so that more learning hours can be provided. The guided learning hours linear algebra in the UK curriculum is about 108 hours (60% textbook content * 180 hours for Further Mathematics A level) but the Hong Kong curriculum only recommends 36 learning hours in linear algebra. Some teachers may even cut the lesson time of the extended parts of Mathematics and the students of the extended parts may be neglected. There is no restricted guidance for the teachers to separate the learning time between the extended part and compulsory part so the teachers should specify 2 parts and arrange the time appropriately. Otherwise, promoting further mathematics education in Hong Kong is futile. If M2 can be considered as an elective, the learning hours can be extended to 184 hours, which is closer to the guided learning hours of Further Mathematics A level in the UK qualifications. The content in linear algebra can be more complete to raise the awareness and the local mathematics level in academics.

Limitation and future direction

The limitations of this study are inadequate research support and a small sample size in interviews and questionnaires. Only a few research papers concern about Hong Kong Secondary Mathematics education development. When comparing with the curriculum in other places, Hong Kong is a really small place but the qualification is internationally significant. The development in Hong Kong education issue deserves to get the attention and be promoted to be internationally recognized. Besides, the majority of the participants in the questionnaire and the researcher have received local mathematics secondary education. The result of this research may conceal bias on one side since the background of the participants are very similar. This research can be further developed if more participants from the UK can be involved to share their views towards their curriculum. The future direction could be comparing local curriculum with those in other countries, such as Singapore, the United State or Canada in different topics of mathematics to lead Hong Kong mathematics curriculum into introspection and substantial improvement. For the recommendation, the feasibility, resource distribution and curriculum reform should be further discussed if the extended part of mathematics could be considered as an elective in Hong Kong.

Conclusion

To conclude, this essay compares the syllabus and textbook content in linear algebra of HKDSE M2 with mathematics qualifications in the UK. The syllabus and textbooks in the UK cover much more topics in vectors and matrices than those in Hong Kong. The differences in topics, representations and textbook characteristics are organized in tables and presented in this research. Moreover, some opinions of both the undergraduates who studied HKDSE M2 and those who have taken CIE or Edexcel IALs mathematics papers are organized in charts. Some recommendations for HKDSE M2 textbooks, syllabus and curriculum are suggested after that for enhancing the teaching contents in linear algebra, for example implementing matrix transformation in the syllabus, replacing some proofs from the textbooks to the supplementary notes and considering the extended part of mathematics as regular electives. However, inadequate research support and small sample size may hinder the development of this research. For further investigation, comparing local curriculum with those in other countries, such as Singapore, the United State or Canada in different topics of mathematics to lead Hong Kong mathematics curriculum into introspection and substantial improvement. More importantly, the awareness of linear algebra is hoped to elevate in local mathematics education and more learning hours and investigations should be implemented in HKDSE M2.

Reference List

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational studies in mathematics*, *52*(3), 215-241.

Armstrong, P. (2016). Bloom’s taxonomy. *Vanderbilt University Center for Teaching*.

Bogomolny, M. (2007). Raising students’ understanding: Linear algebra. In *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 65-72).

Cambridge Assessment International Education. (2020). Cambridge A Level Results Statistics - November 2019

Cambridge Assessment International Education. (2020). Cambridge International AS and A Level Mathematics (9709) 2020-2022 Syllabus.

Cambridge Assessment International Education. (2020). Cambridge International AS and A Level Mathematics - Further (9231) 2020-2022 Syllabus.

Chong, Y. O., Chang, K. Y., Kim, G., Kwon, N. Y., Kim, J. H., Seo, D. Y., ... & Tak, B. (2016). A Comparative Study of Mathematics Curriculum among the United States, Singapore, England, Japan, Australia and Korea. *Journal of Educational Research in Mathematics, 26*(3), 371-402.

Dogan-Dunlap, H. (2010). Linear algebra students’ modes of reasoning: Geometric representations. *Linear algebra and its applications, 432*(8), 2141-2159.

Education Bureau. (2018). Comparison of the Content of Revised Mathematics Curriculum and Current Mathematics Curriculum

Education Bureau. (2018). Explanatory Notes to Senior Secondary Mathematics Curriculum (with updates in August 2018) (Module 2)

Education Bureau. (2018). Explanatory Notes to Senior Secondary Mathematics Curriculum (Module 2)

Hege, H. C., & Polthier, K. (Eds.). (2013). *Visualization and mathematics III*. Springer Science & Business Media.

HKEAA. (2019). 2019 HKDSE Analysis of results of candidates in each subject.

HKEAA. (2019). 2019 HKDSE Entry statistics.

HKEAA. (2019). 2019 HKDSE Statistics overview.

Hooker, S. (2016). *Edexcel AS and A Level Modular Mathematics Mechanics 1 M1*. Pearson Education Limited.

Kolb, A. Y., & Kolb, D. A. (2005). Learning styles and learning spaces: Enhancing experiential learning in higher education. *Academy of management learning & education*, *4*(2), 193-212.

Leung, KS, Hung. FY, Wan, YK, … & Shum, SW. (2015). *New Century Mathematics (2014 Second Edition) Module 2 M2A Algebra and Calculus (Extended Part)*. Oxford University Press (China).

Leung, KS, Hung. FY, Wan, YK, … & Shum, SW. (2015). *New Century Mathematics (2014 Second Edition) Module 2 M2B Algebra and Calculus (Extended Part)*. Oxford University Press (China).

Linsky, J., Nicolson, J., & Western, B. (2015). *Oxford pure mathematics for Cambridge International AS & A level*. Oxford: Oxford University Press.

Pearson Edexcel. (2019). International Advanced Level Mathematics/ Further Mathematics/ Pure Mathematics Specification

Pearson Edexcel. (2020). Grade Statistics - January 2020 (Provisional) - Edexcel International Advanced Level

Pearson Edexcel. (n.d.). Supplementary materials to Hong Kong Diploma of Secondary Education (HKDSE) students preparing for Edexcel International Advanced Level (IAL) exams. Retrieved from https://dev1.pearson.com.hk/index.php?section=473

Penney, R. C. (2008). *Linear algebra: Ideas and applications*. Wiley-Interscience.

Pledger, K. (2016). *Edexcel AS and A Level Modular Mathematics Further Pure Mathematics 1 FP1*. Pearson Education Limited.

Pledger, K. (2016). *Edexcel AS and A Level Modular Mathematics Further Pure Mathematics 3 FP3*. Pearson Education Limited.

Pledger, K. (2017). *Edexcel AS and A Level Modular Mathematics Further Pure Mathematics 2 FP2*. Pearson Education Limited.

Reid, D. A., & Knipping, C. (2010). Proof in mathematics education. *Research, learning and teaching*.

Schaffer, J., & World Scientific. (2015). *Linear algebra*. Singapore; Hackensack, N.J.: World Scientific Pub.

Stewart, S., & Thomas, M. O. (2006). Process-object difficulties in linear algebra: Eigenvalues and eigenvectors. *International Group for the Psychology of Mathematics Education*, 185.

Stewart, S., & Thomas, M. O. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951-961.

Stewart, S., & Thomas, M. O. (2010). Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematical Education in Science and Technology, 41*(2), 173-188.

Thomas, M. O., de Freitas Druck, I., Huillet, D., Ju, M. K., Nardi, E., Rasmussen, C., & Xie, J. (2015). Key mathematical concepts in the transition from secondary school to university. In *The proceedings of the 12th international congress on mathematical education* (pp. 265-284). Springer, Cham.

Tucker, A. (1993). The growing importance of linear algebra in undergraduate mathematics. *The college mathematics journal, 24*(1), 3-9.

Wawro, M., Sweeney, G. F., & Rabin, J. M. (2011). Subspace in linear algebra: Investigating students' concept images and interactions with the formal definition. *Educational Studies in Mathematics, 78*(1), 1-19.

Appendix 1: Interview Protocol

1. Did you take HKDSE M2/ CIE A level / Edexcel IALs mathematics examinations?
2. Do you know about linear algebra? How familiar do you know about linear algebra?
3. Do you know the percentage of linear algebra in the syllabus or textbook content in your examination?

Participants are required to read the comparative tables between the syllabus content in linear algebra in Hong Kong and UK qualifications listed in the research paper P. 39-42 or some Hong Kong textbooks extracts.

4. After reading the information, please provide some comments base on the following criteria:
   a. the syllabus about linear algebra of mathematics paper in Hong Kong and in the UK
   b. the HK and UK textbooks content in linear algebra
5. Do you think that linear algebra can help you in your future? Do you think linear algebra can get enough awareness in your country or living places?
6. Can you provide some recommendations for the secondary mathematics education in Hong Kong?

Appendix 2: Questionnaire

Please tick ✓ in the appropriate category.

Background information:

1 Did you take HKDSE M2 examinations?

☐Yes ☐No

2 Have you heard about UK qualifications?

☐ Cambridge International Examination (CIE)

☐ EDEXCEL International Advanced Levels (Edexcel IALs)

☐ None

Content in linear algebra

3 Please rate your familiarity (from 1 to 5, 5 is the most familiar) with the following criteria in linear algebra:

a. Determinants ( )

b. Matrices ( )

c. Systems of Linear Equations ( )

d. Vectors ( )

4 Please tick the teaching content in linear algebra you have learned:

Vectors	Matrix
☐ find the distance by using the line equation in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$	☐ describe linear transformation by using matrices, such as rotations: about (0,0) of angles that are multiplication of 45° enlargements: centre (0,0) of scalar factor $k(k \neq 0, k \in \mathbf{R})$ reflections: in coordinate axes or the lines $y = \pm x$

	identity: the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is called $\mathbf{I}$ and does not carry out any transformation
☐ displacement vectors $\overrightarrow{AB} = \begin{pmatrix}2\\ -5\\ 1\end{pmatrix} or\ 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$	☐ the inverse of transformation
☐ find the intersection of two lines which are given by vector equations $\mathbf{r} = \begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} + t\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix} + t\begin{pmatrix}d_1\\ d_2\\ d_3\end{pmatrix}$ respectively	☐ solving problems involving invariant lines and points
☐ find the distance from a point to a line $\mathbf{r} = \mathbf{a} + t\mathbf{b}$	☐ singular matrix/ non-singular matrix If $det(\mathbf{A}) = 0$, then $\mathbf{A}$ is a singular matrix and $\mathbf{A}^{-1}$ cannot be found. If $det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is a non-singular matrix and $\mathbf{A}^{-1}$ exists.
☐ volume of tetrahedron $\left\lvert \frac{1}{6}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \right\rvert$	☐ property of transpose of a matrix
☐ line equation in the form of $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$	☐ minor of a matrix
☐ general Cartesian equation of a straight line is $\frac{(x - x_1)}{l} = \frac{(y - y_1)}{m} = \frac{(z - z_1)}{n} = \lambda,$	☐ using determinants of a matrix to determine the area scalar factor of the transformation **Area of image**

where the lines passes through the point (x_1, y_1, z_1), has direction ratios $l\!:\!m\!:\!n$, and where λ is a parameter.	$= \textbf{Area of object} \times \lvert \boldsymbol{det}(\mathbf{M}) \rvert$
☐ converting equations of planes into different forms: $ax + by + xz = d$, $\mathbf{r} \cdot \mathbf{n} = p$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$	☐ characteristic equation of $\mathbf{A}$ is the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
☐ angles between line and plane $\sin\theta = \left\lvert \frac{\mathbf{b} \cdot \mathbf{n}}{\lvert\mathbf{b}\rvert\lvert\mathbf{n}\rvert} \right\rvert$	☐ An eigenvector of a matrix $\mathbf{A}$ is a non-zero column vector $\mathbf{x}$ which satisfies the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, where λ is a scalar which is the corresponding eigenvalue of the matrix.
☐ angles between 2 planes $\cos\theta = \left\lvert \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{\lvert\mathbf{n_1}\rvert\lvert\mathbf{n_2}\rvert} \right\rvert$	☐ If $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is an eigenvector of a matrix, then the corresponding normalised eigenvector is $\begin{pmatrix} \frac{a}{\sqrt{(a^2+b^2+c^2)}} \\ \frac{b}{\sqrt{(a^2+b^2+c^2)}} \\ \frac{c}{\sqrt{(a^2+b^2+c^2)}} \end{pmatrix}$.
☐ shortest distance between 2 skew lines with equations $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$, where λ and μ are scalars, is given by the formula $\left\lvert \frac{(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d})}{\lvert\mathbf{b} \times \mathbf{d}\rvert} \right\rvert$	☐ If $\mathbf{M}$ is an orthogonal matrix consisting of the normalized column vectors $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$, then $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_3 = \mathbf{x}_3 \cdot \mathbf{x}_1 = 0$.
☐ vector application to displacements, accelerations, forces	☐ diagonal matrix: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}$.
☐ If a particle starts from the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, then its displacement from its initial position at	☐ reduction of symmetric matrices to diagonal form: When symmetric matrix $\mathbf{A}$ is reduced to a diagonal matrix $\mathbf{D}$,

time t is $\mathbf{v}t$ and its position vector $\mathbf{r}$ is given by $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$.	the elements on the diagonal are the eigenvalue of $\mathbf{A}$.
☐ If a particle with initial velocity $\mathbf{u}$ moves with constant acceleration $\mathbf{a}$ then its velocity, $\mathbf{v}$, at time t is given by $\mathbf{v} = \mathbf{u} + \mathbf{a}t$.	
☐ The force causes the particle to accelerate: $\mathbf{F} = m\mathbf{a}$, where m is the mass of the particle.	

5 If you can tick any of the above boxes, where can you access the above content in linear algebra?

☐ From HKDSE M2 textbooks
☐ From courses in universities
☐ Self-study
☐ Others

Recommendation towards Hong Kong secondary mathematics education or textbooks

6 Do you think linear algebra can help you in your future?

☐ Yes, please specify: future careers/ study/ or others: __________
☐ No

7 Do you think linear algebra can get enough awareness in your country or living places?

☐ Yes ☐ No

8 Recommendations for the secondary mathematics education or textbooks in Hong Kong

____

End

Thank you for your participation!

Appendix 3: Consent Form of Participants

THE EDUCATION UNIVERSITY OF HONG KONG

Department of Mathematics and Information Technology

CONSENT TO PARTICIPATE IN RESEARCH

Comparing the UK advanced level contents and M2 contents in linear algebra

I ____________________ hereby consent to participate in the captioned research supervised by Dr Yuen Man Wai and conducted by Chiu Hing Lun, who are staff / students of Department of Mathematics and Information Technology in The Education University of Hong Kong.

I understand that information obtained from this research may be used in future research and may be published. However, my right to privacy will be retained, i.e., my personal details will not be revealed.

The procedure as set out in the **attached** information sheet has been fully explained. I understand the benefits and risks involved. My participation in the project is voluntary.

I acknowledge that I have the right to question any part of the procedure and can withdraw at any time without negative consequences.

Name of participant ______________________________

Signature of participant ______________________________

Date ______________________________

INFORMATION SHEET

Comparing the UK advanced level contents and M2 contents in linear algebra

You are invited to participate in a project supervised by Dr Yuen Man Wai and conducted by Chiu Hing Lun, who are staff / students of the Department of Mathematics and Information Technology in The Education University of Hong Kong.

The introduction of the research

The research aims at comparing the contents of HKDSE Module 2 and the Pure Mathematics and Further Pure Mathematics parts of UK qualifications in linear algebra. Some opinions of both the undergraduates who took HKDSE M2 exam and those who took UK qualifications mathematics papers will be collected. Through the comparison and participants' comments, some recommendations can be suggested to improve the secondary Mathematics education in Hong Kong.

The methodology of the research

The participants of this study are separated into 2 groups, the local undergraduates who took HKDSE M2 and those who took UK qualifications (CIE A level or Edexcel IALs) math papers before. For qualitative methods, interviews are implemented for 3 randomly selected participants of each group in order to deeply understand their views toward UK qualifications and HKDSE M2. Some documents about the syllabus of HKDSE M2 or UK qualifications mathematics content would be prepared and distributed in the interview for students to know the syllabus of both examinations and give their comments on them. The interview participants can get HK$ 67/ hours for 1 hour interview. The participants are required to read the syllabus and textbook contents of HKDSE M2 and UK qualifications. So, they can share their views towards both examinations. For the quantitative methods, 50 questionnaires can be distributed for the local undergraduates who took HKDSE M2 in order to assess their knowledge about linear algebra and collect their comments towards the content of CIE.

The potential risks of the research (State explicitly if none)

The study involves no potential risk.

Your participation in the project is voluntary. You have every right to withdraw from the study at any time without negative consequences. All information related to you will remain confidential, and will be identifiable by codes known only to the researcher.

Describe how results will be potentially disseminated

This research will be published in the form of thesis and educational presentations.

If you would like to obtain more information about this study, please contact Chiu Hing Lun at telephone number or their supervisor Dr Yuen Man Wai at telephone number

If you have any concerns about the conduct of this research study, please do not hesitate to contact the Human Research Ethics Committee by email at hrec@eduhk.hk or by mail to Research and Development Office, The Education University of Hong Kong.

Thank you for your interest in participating in this study.

Chiu Hing Lun
Principal Investigator

Appendix 4: Transcripts of interviews

<u>Interviewee 1: Kelly Tang</u>

Background information:

She is an undergraduate of one of the oversea universities. She took CIE Pure Mathematics paper before.

Interviewer: Halo, Kelly. I am going to do a project to compare the UK advanced level contents and M2 contents in linear algebra. I know that you took CIE Pure Mathematics before. Do you know about linear algebra? How familiar do you know about linear algebra?

Interviewee: I am not so familiar with it and my result was not so good. What chapters do you mean?

Interviewer: Linear algebra include the contents of vectors and matrix. Could you remember anything about them?

Interviewee: Oh, I got it. I learnt vectors and I think it is quite difficult. I think I may get the lowest marks in this chapter since we did not study it before the high school. I can only barely remember it. We learned differentiation and integration in most of the lesson time.

Interviewer: So you only learned 1 chapter about vectors in your high school?

Interviewee: Yes, I think.

Interviewer: So now I am going to give you some comparative tables between the syllabus content in linear algebra in Hong Kong and UK qualifications listed in the research paper and some extracts from Hong Kong textbooks. Please give me some comments about the syllabus or textbooks content about linear algebra of mathematics paper in Hong Kong and in the UK

Interviewee: Ok

After 15 minutes,

Interviewer: Can you give the comments now?

Interviewee: I can remember more about linear algebra after seeing these. I can see that the syllabus of UK qualifications covers much more topics in linear algebra than Hong Kong. So Hong Kong syllabus should be easier for me. But Hong Kong textbook is so long for me to read and it is quite boring. (There are) Many questions and exercises behind. The UK textbooks show fewer proofs but more diagrams for us. I like UK (textbooks) more.

Interviewer: Thank you for your comments. Do you think linear algebra can help you in your future? Do you think linear algebra can get enough awareness in your country or living places?

Interviewee: I don’t think it is beneficial for me since I am not studying in math now and my syllabus only cover one chapter of linear algebra. More contents in linear algebra should be included since I am now feeling interested in it. I am willing to learn more and know the difference between us and the Hong Kong students.

Interviewer: It’s grateful to hear that. Can you provide some recommendations for the secondary mathematics education in Hong Kong? Or the Hong Kong textbooks?

Interviewee: I think adding more interesting diagrams can let students to easily understand and not getting bored in the lesson. More lesson time is needed for the Hong Kong students to learn the linear algebra.

Interviewer: Thank you so much for your comments and let’s keep in touch.

Interviewee: Thank you.

End

Interviewee 2: Adrian Tse

Background information: He is studying engineering in the universities in UK. He took Pure mathematics and Mechanics in Edexcel IALs.

Interviewer: Halo, Adrian. I am going to do a project to compare the UK advanced level contents and M2 contents in linear algebra. I know that you took Pure mathematics and Mechanics in Edexcel IALs before. Do you know about linear algebra? How familiar do you know about linear algebra?

Interviewee: I am pretty good in vectors. Since it is included in the Pure Math and Mechanics, and I like physics. So it is interesting when it can applied in the mechanics like displacements, accelerations, forces. It is pretty easy.

Interviewer: Ok. Then I am going to give you some comparative tables between the syllabus content in linear algebra in Hong Kong and UK qualifications listed in the research paper and some extracts of Hong Kong textbooks. Please give me some comments about the syllabus or textbooks content about linear algebra of mathematics paper in Hong Kong and in the UK

Interviewee: It's good.

After 15 minutes,

Interviewee: Wow! So many differences in the syllabus and textbooks between the UK and Hong Kong. All of them are covered in linear algebra? Some of them I don't even learned in my high school.

Interviewer: There are some contents in the Further Mathematics. Matrix is one of them.

Interviewee: I see. It's a really good work. I really like vectors and mechanics. I don't think it is difficult to apply vectors in physics but it seems that Hong Kong syllabus doesn't cover the application. It is hard to learn if it cannot be applied in the real situation. And Hong Kong textbooks teach matrix first and then vectors! I wish I can study in Hong Kong to learn matrix.

Interviewer: Are you interested in matrix? You can take Further Pure Math to learn it.

Interviewee: Yeah, but I don't want to take Further Pure Math. I like linear algebra and it is one of the most important topics in engineering in my university.

Interviewer: Ok. Do you think linear algebra can help you in your future? Do you think linear algebra can get enough awareness in your country or living places?

Interviewee: It can definitely be applied in mechanics and it can help for my further study or careers. I'm interested in linear algebra but what my syllabus covers is not enough so I don't think the awareness of it is high enough.

Interviewer: Thank you for your comments and can you provide some recommendations for the secondary mathematics education in Hong Kong? Or the Hong Kong textbooks?

Interviewee: More contents relating to vectors should be added for the Hong Kong students. It is good for them to know that vectors can be applied in different situations. They can also learn more about mechanics as well.

Interviewer: Thank you Adrian and keep in touch.

Interviewee: Ok see you.

End

Interviewee 3: Janice Chan

Background information: He is studying mathematics in the universities in UK. He took Pure mathematics and Further Mathematics in Edexcel IALs.

Interviewer: Halo, Janice. I am going to do a project to compare the UK advanced level contents and M2 contents in linear algebra. I know that you took Pure mathematics and Further Mathematics in Edexcel IALs before. Do you know about linear algebra? How familiar do you know about linear algebra?

Interviewee: I enjoy learning linear algebra and it's pretty easy in the syllabus. It is good for me to learn vectors and matrices in high school and I can have a clear mind about linear algebra when I study in the universities.

Interviewer: Very nice. Then I am going to give you some comparative tables between the syllabus content in linear algebra in Hong Kong and UK qualifications listed in the research paper and some extracts of Hong Kong textbooks. Please give me some comments about the syllabus or textbooks content about linear algebra of mathematics paper in Hong Kong and in the UK

Interviewee: Ok thank you.

After 15 minutes,

Interviewee: There are many missing linear algebra topics in the Hong Kong syllabus, like eigenvectors, matrix transformation and vectors equations of line and planes. One topic called solving systems of linear equations is special in Hong Kong syllabus. The UK textbook doesn't spend a chapter to introduce it. Also, some notification of vectors and matrices are different in two textbooks. The introduction of displacement vectors is one of the examples.

Interviewer: It's good for you to see so many details of those differences. Thank you. Do you think linear algebra can help you in your future? Do you think linear algebra can get enough awareness in your country or living places?

Interviewee: I think it is good for my future study and there is more I can explore in it. It's pretty interesting. Further Pure Mathematics covers more about linear algebra. At least half of the curriculum is related to it and I'm glad to choose this subject.

Interviewer: Thank you for your comments and can you provide some recommendations for the secondary mathematics education in Hong Kong? Or the Hong Kong textbooks?

Interviewee: The level of linear algebra of Hong Kong students should be improved. More contents and teaching hours should be added. It's not enough for the students who are really interested in mathematics and willing to learn more.

Interviewer: Thank you Janice and keep in touch.

Interviewee: Thank you, bye!

End

Interviewee 4: Kaman Lo

Background information: She is studying mathematics in one of the universities in Hong Kong. She took HKDSE M2 before.

Interviewer: Halo, Kaman. I am going to do a project to compare the UK advanced level contents and M2 contents in linear algebra. I know that you took HKDSE M2 before. Do you know about linear algebra? How familiar do you know about linear algebra?

Interviewee: Yes. I know that covers vectors and matrices. I can still what I have learnt in my secondary school.

Interviewer: Nice. Then I am going to give you some comparative tables between the syllabus content in linear algebra in Hong Kong and UK qualifications listed in the research paper and some extracts of UK textbooks. Please give me some comments about the syllabus or textbooks content about linear algebra of mathematics paper in Hong Kong and in the UK

Interviewee: Ok. I got it.

After 15 minutes,

Interviewee: All of these are the contents in linear algebra missing in the HKDSE M2? It surprised me a lot. Some of them are taught in the university's courses. The UK students have to learn all of them in high school? In the UK textbooks, there are many examples about the application of the theories and less proofs are included. Don't they need to learn the proofs of the theories?

Interviewer: Which textbook do you like more?

Interviewee: The HK textbooks. Since the proofs can let us to develop our skills of proving mathematics theories and we can know how they can be developed.

Interviewer: Do you think linear algebra can help you in your future? Do you think linear algebra can get enough awareness in your country or living places?

Interviewee: Study the linear algebra in secondary school is good to build up the fundamental for future study in mathematics. And I think linear algebra can get enough awareness in M2 syllabus.

Interviewer: Ok. Can you provide some recommendations for the secondary mathematics education in Hong Kong? Or the Hong Kong textbooks?

Interviewee: Teachers in my secondary school taught both compulsory mathematics and M2 together in a double lesson. Sometimes it's quite confused and rash in the teaching content. There should be a clearer separation. M2 should be set up like the Further Mathematics in UK syllabus so that the teaching time in linear algebra can be lengthened and more contents can be completed.

Interviewer: Thank you Kaman and keep in touch.

Interviewee: OK. Bye.

End

Interviewee 5: Kelsey Chou

Background information: She is studying social science in one of the universities in Hong Kong. She took HKDSE M2 before.

Interviewer: Halo, Kelsey. I am going to do a project to compare the UK advanced level contents and M2 contents in linear algebra. I know that you took HKDSE M2 before. Do you know about linear algebra? How familiar do you know about linear algebra?

Interviewee: Not really. What do you mean?

Interviewer: Did you learn vectors and matrices in your DSEM2 curriculum? They are a part of linear algebra.

Interviewee: Oh, yes. But I cannot remember those anymore. I'm not so good in math.

Interviewer: Are they difficult? How do you feel when you were studying them?

Interviewee: It is an awful experience for me. I couldn't understand what the teacher taught in the lesson. When I tried to ask my teacher or friends, they're very busy to handle their own work. The teacher just told us to complete the classwork by apply the skills shown in the exercise. They had no time to teach M2 and mathematics together.

Interviewer: Ok. Then I am going to give you some comparative tables between the syllabus content in linear algebra in Hong Kong and UK qualifications listed in the research paper and some extracts of UK textbooks. Please give me some comments about the syllabus or textbooks content about linear algebra of mathematics paper in Hong Kong and in the UK

Interviewee: Ok.

After 15 minutes,

Interviewer: Could you give your comments now?

Interviewee: Yes, but I am sorry that I haven't learned a lot on the list. But when I look at the UK textbooks, it is much clearer and shorter than my textbooks. I don’t like the long proofs of the theories since it is not a must-read material and it won’t be covered the exam. I usually skip it and copy the examples to do the exercise on my own. They can’t help me to learn. Doing so many drillings is too exhausting and I don’t have enough time to practice and discuss with my friend.

Interviewer: Do you think linear algebra can help you in your future? Do you think linear algebra can get enough awareness in your country or living places?

Interviewee: I don’t think linear algebra can help me in my life anymore. I am not studying math and going to study. I almost forgot I have learned something about vectors or matrix and there are only very few M2 students in Hong Kong.

Interviewer: Ok. Can you provide some recommendations for the secondary mathematics education in Hong Kong? Or the Hong Kong textbooks?

Interviewee: If I could learn the UK textbooks, I think I could be better in learning linear algebra. The HK textbooks should reduce the proofs. The authorities should consider M2 as one of the regular electives but it is really hard.

Interviewer: Thank you Kelsey and keep in touch.

Interviewee: OK. Bye.

End

Interviewee 6: Hugo Chui

Background information: He is studying in science major program in local universities. He took HKDSE M2 before.

Interviewer: Halo, Hugo. I am going to do a project to compare the UK advanced level contents and M2 contents in linear algebra. I know that you took HKDSE M2 before. Do you know about linear algebra? How familiar do you know about linear algebra?

Interviewee: The vectors and the matrix stuff. I can still remember. They are different from the other topics, like the differentiation or integration.

Interviewer: It's good. Then I am going to give you some comparative tables between the syllabus content in linear algebra in Hong Kong and UK qualifications listed in the research paper and some extracts of UK textbooks. Please give me some comments about the syllabus or textbooks content about linear algebra of mathematics paper in Hong Kong and in the UK

Interviewee: Ok.

After 15 minutes,

Interviewee: It is impressive to find out all those difference between 2 countries math syllabus and textbooks. In the beginning, I thought the UK textbooks must be much more difficult to understand and thicker than our textbooks. But it doesn't look like this. I like the UK textbooks style. It's so easy to understand and the instructions are clear. It's funny in the beginning part and then I can remember and think more. I don't like to remember those stupid properties and theories (in HK textbooks).

Interviewer: Do you find some new knowledge for you?

Interviewee: I don't know matrix transformation can be drawn on the graph paper. This representation is funny.

Interviewer: Do you think linear algebra can help you in your future? Do you think linear algebra can get enough awareness in your country or living places?

Interviewee: Learning more must be better for my study. Learning linear algebra can help me to pass the mathematics courses in the university. I will try to find the textbooks in UK when I take math course. Thank you.

Interviewer: Nice. Can you provide some recommendations for the secondary mathematics education in Hong Kong? Or the Hong Kong textbooks?

Interviewee: Less proofs and more diagrams are preferred in the HK textbooks. Adding more funny diagrams and examples must be better to raise my interest.

Interviewer: Thank you Hugo and keep in touch.

Interviewee: OK. Bye.

End