Contributing Factors for Improving Mathematics Teaching for Students with Intellectual Disabilities

by

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Statement of Originality

I, WANG, Dichen, hereby declare that I am the sole author of the thesis and the material presented in this thesis is my original work except those indicated in the acknowledgement. I further declare that I have followed the University's policies and regulations on Academic Honesty, Copyright and Plagiarism in writing the thesis and no material in this thesis has been submitted for a degree in this or other universities.

Abstract

Subject-specific training in mathematics teaching for students with intellectual disabilities (IDs) has been little studied in the academic field and is uncommon in the practice of teacher education. Given that teachers face significant challenges in regard to teaching mathematics to ID students, it is necessary to know whether there is a need for subject-specific training. If yes, what kind of subject-specific training do teachers need? What kind of subject-specific knowledge should be covered? How can this kind of training contribute to improving mathematics teaching for ID students?

To answer the above questions, this study focuses on one particular teacher professional development programme (BE MATHS programme) for mathematics teachers who teach ID students in special schools in Hong Kong. Programme effects and the contributing factors of the effects are examined to explore whether and how subject-specific support can contribute to the profession of teaching mathematics to ID students.

The study is conducted via a mixed-methods approach. First, it uses quantitative measures to examine teachers' changes in mathematics teaching efficacy and students' changes in their academic engaged time in mathematics classes. Second, a qualitative approach is used to explore the contributing factors of the changes.

Quantitative results show that the programme has a significant positive effect on teachers' mathematics teaching outcome expectancy and a conditional positive effect on teachers' personal mathematics teaching efficacy. It is also found that the

programme can improve teachers' mathematics teaching by engaging more students in mathematics learning.

The qualitative findings in this study show that the design of teaching and learning trajectories with subject experts is a substantial learning activity for teachers. During the design process, teachers and subject experts identified knowledge gaps within the mathematical content to be delivered to ID students and successfully inserted intermediate learning stages to bridge the gaps. Three case studies conducted in schools for students with mild, moderate, and severe IDs respectively provide considerable evidence on the existence of specialised content knowledge that is essential to teaching mathematics to ID students. The findings highlight the need to study mathematics teaching problems in special schools from a subject-based perspective and accumulate specialised content knowledge to prepare mathematics teachers for special education and inclusive education.

In summary, the study indicates that pedagogical knowledge and a superficial understanding of mathematics (compared with profound understanding of fundamental mathematics) are not enough for teaching mathematics to ID students. The profession of teaching mathematics to ID students can be improved by studying the teaching and learning trajectories of ID students from a subject-specific perspective via a design research approach. The findings have practical implications for professional developers and address how teachers can be better prepared to develop their profession on ID students' mathematics teaching. This is a significant contribution, given that subject-specific knowledge is lacking by teachers in special schools but has little been addressed in teacher training programmes and academic research fields.

Keywords: Design research, special education, teacher education, Mathematics instruction, students with intellectual disabilities, BE MATHS programme

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ADHD	Attention-deficit/hyperactivity disorder
ASD	Autism spectrum disorders
CCK	Common content knowledge
CK	Content knowledge
CTPD	Characteristics of Teacher Professional Development
GPK	General pedagogical knowledge
HI	Hearing impairment
ID	Intellectual disability
KC	Knowledge of curriculum
KCS	Knowledge of content and students
KCT	Knowledge of content and teaching
MTEBI	Mathematics Teaching Efficacy Belief Instrument
MTOE	Mathematics teaching outcome expectancy
PCK	Pedagogical content knowledge
PD	Physical disabilities
PMTE	Personal mathematics teaching efficacy
RQ	Research question
SCK	Specialised content knowledge
SEN	Special educational needs
SMK	Subject matter knowledge
SpLD	Specific learning difficulties
VI	Visual impairment

Chapter 1 Introduction

1.1 Contextual Background

In recent decades, many countries and states have made significant efforts to provide quality education for all children (United Nations Educational Scientific and Cultural Organisation [UNESCO], 2015). In addition to reducing inequalities in education that originate in gender and cultural identities, nations have also given increasing attention to providing high-quality education for students with diverse special educational needs (SEN). SEN students refer to those students who experience difficulties or disabilities that make it significantly harder for them to learn compared to other learners of the same age. These difficulties include specific learning difficulties (SpLD), intellectual disabilities (ID), attention-deficit/hyperactivity disorder (ADHD), visual impairment (VI), hearing impairment (HI), physical disabilities (PD), and autism spectrum disorders (ASD).

In Hong Kong, SEN students account for approximately 8.15% of the student population (Finance Committee, 2017). According to local policies, while all SEN students can attend mainstream schools, some SEN students can be referred to special schools for intensive support on the recommendation of specialists and with their parents' consent. Both special schools and mainstream schools follow the same curriculum framework, which means that all SEN students—regardless of the type and severity of their disabilities—have opportunities to access all the subjects and knowledge offered to mainstream students.

Under the current education system, mathematics teachers of ID students always suffer from a difficult situation because of the lack of support they face in regard to subject-specific knowledge. Intelligence is a general mental ability that involves the capability "to reason, plan, solve problems, think abstractly, comprehend complex ideas, learn quickly, and learn from experience" (Gottfredson, 1997, p. 13). The American Association on Intellectual and Developmental Disabilities (2010) defines ID as a disability that is "characterised by significant limitations both in intellectual functioning and in an adaptive behaviour as expressed in conceptual, social, and practical adaptive skills. … [which] originates before age 18", thereby distinguishing children with ID from the general population. Children with ID face significant obstacles regarding cognitive issues, academic achievement, social competence, and communication, including difficulty with abstract thinking and comprehension, weak listening skills, poor short-term memory, and taking an extraordinarily long time to learn (Taylor, 2005). However, mathematics is an activity that is full of abstraction and logical thinking, which puts intensive pressure on students' working memory, long-term memory, and information processing ability (Barrouillet & Lépine, 2005; Swanson et al., 2008). Since ID students have weaker memory and a considerably lower speed of information processing (Pickering & Gathercole, 2004), general methods of mathematics teaching for students with normal intelligence may be ineffective for them. There is therefore a critical need among teachers for additional knowledge of practical mathematics learning trajectories specifically designed for ID students.

However, few training programmes have been established in Hong Kong to help teachers cater to ID students' disabilities in mathematics education. Inclusive education training has been launched in preservice teacher education. The Education Bureau

(2017c) has also launched basic, advanced, and thematic (BAT) courses (ranging from 30 to 252 hours) for in-service teachers to support them when they are faced with the significant challenges of providing inclusive education and special education. However, most of these courses are designed only for the generic aspects of SEN education, in which psychology-based knowledge constitutes a substantial part, whereas subject-specific training (such as for mathematics and science) is virtually missing. When discussing the role of psychology in mathematics education, Polya (1963), who is one of the most influential mathematicians and mathematics educators of the 20th century, pointed out that while the psychology of learning may provide teachers and educators with interesting hints, "it cannot pretend to pass ultimate judgement upon problems of teaching" (p. 605). Although psychological findings regarding children's learning disabilities may inform teachers about the sources of students' difficulties in learning mathematics, they cannot provide teachers with solutions for overcoming difficulties in teaching. The problem of how to teach mathematical content given the constraints of students' disabilities remains open and unresolved.

A lack of resources is another challenge for teachers when teaching mathematics to ID students. Within the long list of over 40 learning and teaching resources recommended by the Education Bureau (2017a), only one item is designed for students with learning difficulties, and it is not specifically designed for ID students. Unlike the resources available regarding students in mainstream schools, the teaching resources that target ID students are rare. It follows that mathematics teachers in special schools are suffering an immense shortage of teaching resources. Consequently, teachers' instruction is heavily driven by their own experiences in learning mathematics. As expected, with a growth in the quantity of practical classroom experiences, teachers soon come to realise that the methods they learned about teaching mathematics before do not address the learning needs of ID students. With limited subject-specific training and few teaching resources, mathematics teachers of ID students have to tackle problems related to teaching without support from the rest of the educational field.

The problem of neglecting subject-specific support when addressing the needs of ID students and other SEN students in academic learning is global. Although the number of studies on mathematics education for SEN children has increased (Marita & Hord, 2017; Myers et al., 2015), studies with an emphasis on teaching mathematics to SEN students are seldom found (Lambert & Tan, 2016). Likewise, while many studies have investigated the features of effective teacher education programmes, their findings only inform the training of general education teachers. Upon conducting a contemporary synthesis of relevant studies, Allsopp and Haley (2015) found that from 2004 to 2014, only 16 studies included criteria for teacher education, mathematics, and SEN students. Only one of these 16 studies included students in special schools, with no specific information regarding the particular disabilities of students. The authors acknowledged a pressing need to identify factors that contribute to the adequate training of teachers teaching mathematics to SEN students, in particular ID students.

Is subject-specific training in mathematics teaching for ID students necessary? If so, what does it look like?

1.2 The BE MATHS Programme

There are currently 61 special schools in Hong Kong, of which 41 are for ID students (Education Bureau, 2017b).

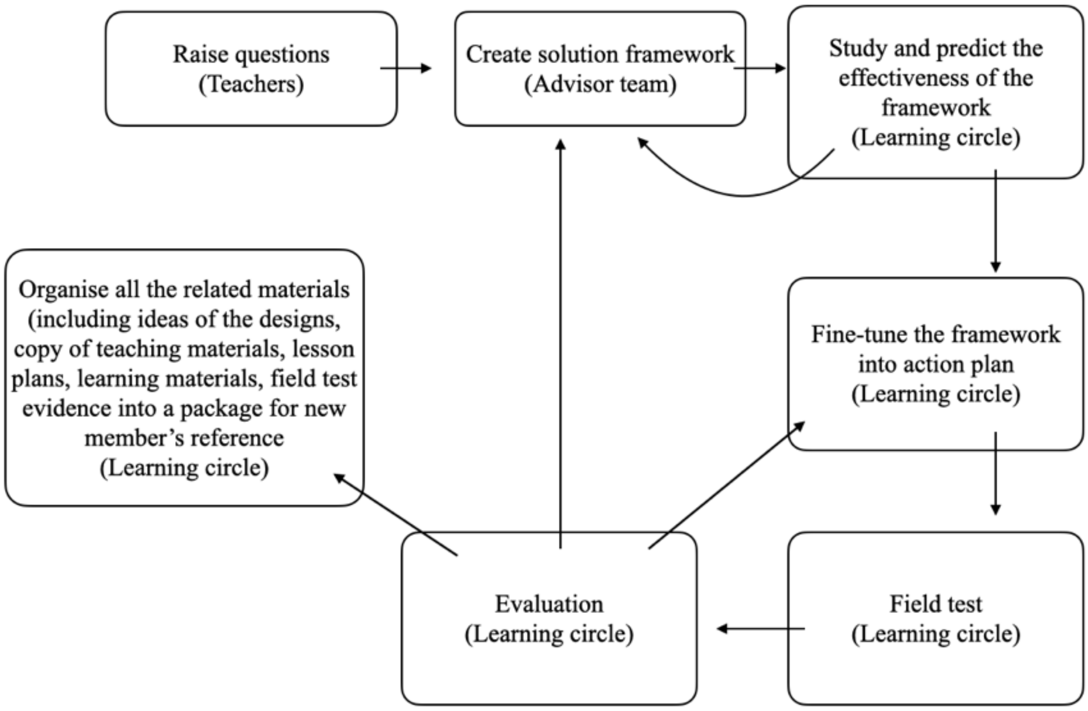
Launched in 2014, the BE MATHS programme is a professional development (PD) programme for implementing the mathematics curriculum under basic education for ID students. Ten to twelve Hong Kong special schools with ID students (ages 7–18) join the programme each school year.

Learning Circles

Teachers interested in investigating the same mathematics topics form a learning circle. When teachers apply for the programme, they submit an information sheet stating the topics or issues they would like to explore in the programme. Based on these sheets, teachers with the same interests and a subject advisor are grouped into a learning circle, where they share their knowledge and experience, learn new information, and test their ideas together. All the participants are acknowledged that there is no hierarchical structure in the learning circle and that all members meet as equals.

Members of a learning circle meet regularly throughout the programme. They conduct a series of discussions, demonstrations, and reflections, through which they exchange teaching ideas and materials, as well as share practical experiences about teaching mathematics to ID students. During the meetings, members collaborative plan lessons and share knowledge about their students. They also reflect on, review, and revise school-based curricula and teaching materials. All the learning circles are required to conduct teaching experiments to test the ideas and materials constructed during meetings. Figure 1.1 shows the problem-solving action flow of a learning circle.

Figure 1.1
Problem-Solving Action Flow in a BE MATHS Programme Learning Circle



<u>Advisory Team</u>

The advisory team of the programme comprises four advisors. Three are expert teachers in regard to teaching elementary mathematics, and the other is a teacher educator with over 20 years of teacher education experience, who together represent a profound level of knowledge about mathematics and its teaching. Moreover, the teacher educator has authored nearly one hundred related publications, while the three teachers have authored 3 to 6 publications each. The team has worked together on providing support to teachers teaching ID students since 2014. For the programme, the team takes Freudenthal's notion of mathematising as their guide to study mathematics and its teaching. The teacher educator manages the analysis of mathematical knowledge and its mathematising process. The other three advisors take charge of the school-based support activities, where they serve as facilitators for using the knowledge and resources developed through the programme.

Professional Development Activities

The programme begins with a two-and-a-half-day-long centralised meeting at the beginning of the school year. It includes a briefing session for the project, followed by workshops that familiarise participating teachers with the rationale of teaching for mathematising, its implementation in special schools, the resources developed in the programme, and knowledge of developing a professional learning community. Approximately 12 meetings are arranged for each of the learning circles. School teachers conduct these meetings with support from programme advisors. After designing the teaching units with other members in the learning circle, participating teachers try out their teaching ideas in the classroom throughout the school year, while other members conduct peer lesson observations and post-lesson discussions. At the end of the school year, a one-day experience sharing session is organised to showcase the practices of participating teachers.

1.3 Outline of This Thesis

In view of the contextual needs, the availability of a mathematics-oriented professional development programme for supporting special education teachers is uncommon worldwide and rare in Hong Kong. This study therefore takes great interest in exploring the changes that teachers experience through participating in the BE MATHS programme and the contributing factors to these changes.

The rest of this thesis is organised as follows. Chapter 2 presents a literature review on professional development for teaching ID students. Chapter 3 then describes the research questions, research design, methodologies, and procedures that were conducted to answer the research questions. Chapters 4 and 5 report the results of questionnaire analyses, observations of students' academic engaged time, and findings generated from qualitative data, respectively. Chapter 6 summarises and discusses the findings of the research, considers its significance and practical implications, and proposes directions for future research.

Chapter 2 Literature Review

In this section, the literature review provides the theoretical underpinnings for this study, including the need for teaching mathematics to ID students, mathematics education for ID students, mathematics teachers' professional knowledge, teacher training approaches, a conceptual framework for studying professional development, teachers' mathematics teaching efficacy, and student academic engaged time.

2.1 The Needs for Teaching Mathematics to ID Students

In discussions of education for ID students, one controversial issue has been whether ID students need to learn academic subjects at all (Ayres et al., 2011; Courtade et al., 2012). Some researchers have questioned the need for ID students to learn academic subjects such as mathematics and science. They have argued that teaching functional skills directly linked to ID students' daily lives (e.g., working, housing, shopping, and communication) would be more appropriate (Ayres et al., 2011). Others have considered that teaching only functional skills and no academic subject knowledge at all to ID students goes against the notion of education for all, with one of its goals being giving students the right to full educational opportunities (Göransson et al., 2015). The findings of past studies have shown that the full potential of ID students has not yet been recognised and that increases in ID students' academic achievement have also caused educators' expectations to rise continuously (Browder et al., 2012). Therefore, teaching less academic content to ID students is, at best, disputable.

The current research is based on the belief that all students have a right to learn mathematics for three reasons:

1. ID students can learn mathematics if their teachers use appropriate pedagogical approaches (Bashash et al., 2003; Browder et al., 2012; Chung & Tam, 2005; Göransson et al., 2015; Hord & Bouck, 2012; Jimenez & Staples, 2015);
2. ID students' competence in mathematical knowledge (e.g., number concepts, time measurement, and data analysis) can help them master functional skills such as paying bills and managing time; and
3. Mathematical competence can help them gain better employment (Benz et al., 1999; Parmenter, 2011) and live more independently (Eggleton et al., 1999).

2.2 Mathematics Education for ID Students

In recent decades, researchers have conducted several systematic reviews on interventions in mathematics education for ID students. Reviewing intervention studies in mathematics education for students with mild-to-moderate ID, Butler et al. (2001) found 16 papers published from 1989 to 1998. Similarly, reviewing the literature for mild ID students from 1999 to 2010, Hord and Bouck (2012) found seven relevant studies. Last, for their comprehensive review of publications from 1975 to 2005 on intervention studies for individuals with significant ID, including students with moderate-to-severe ID, Browder et al. (2008) found 45 studies including moderate ID students and 17 studies including severe ID students. Although the three reviews covered different periods and participants with different ID levels, they all reported two similar findings:

1. The instruction of mathematics to ID students focuses on direct instruction, where students have little initiative in thinking and cognitive construction; and
2. Regardless of the quality of study, the type of instruction and students' ID level, the results of teaching mathematics to ID students are generally positive.

Although a change in the research paradigm focusing on students' conceptual understanding of mathematics has been carried out in recent decades (Göransson et al., 2015), the instructional methods for teaching students with ID and other learning disabilities have not changed much. A recent meta-analysis has shown that direct instruction is still frequently used to teach students with ID or mathematical learning difficulties (Scherer et al., 2016). While certain researchers have claimed that direct instruction is the most effective teaching approach for ID students (Browder et al., 2008; Kauffman & Hung, 2009), others have criticised that this kind of education regards mathematics as a ready-made product, in which the students simply substitute numerical values into formulas and calculate results. As a result, students have little initiative in regard to thinking and cognitive construction (Dörfler & McLone, 1986), which ultimately leads to inequalities in mathematics education.

Every student has the right to understand mathematics (National Council of Teachers of Mathematics, 2000). However, "how much students with learning disabilities will learn depends on how they are taught" (Browder, 2015). Research has shown that ID children can benefit from conceptual mathematics instruction. Chung and Tam (2005) compared the effects of three instructional methods on the ability of students with mild ID to solve mathematical problems:

1. Conventional instruction, in which students read the problem, write the equation down, and perform the computation;
2. Worked example instruction, in which students visualise the problem and study the worked solutions; and
3. Cognitive strategy instruction, in which students read the problem, paraphrase and visualise it, then write the equation down, compute an answer, and check it.

Their results showed that students from the worked example group and the cognitive strategy group solved more problems than those from the conventional instruction group, both in the immediate test and the 14-day delayed test. The investigators suggested that worked example instruction and cognitive strategy instruction taught ID students to translate a problem from words to pictures, which helped them understand its structure and thus engage in problem-solving activities. Students in both groups could therefore recognise the connections between similar problems and apply the skills learned in class to solve them. In contrast, ID students taught using conventional instruction through rote practice focused on the memorisation of problem features. As a result, those students could not develop problem-solving strategies in class and apply them to other problems. (Chung & Tam, 2005)

Likewise, Göransson et al. (2015) highlighted that teaching for conceptual understanding could help engage ID students in discussions about mathematical topics and improve their reasoning ability. During the mathematics classes they observed, Göransson et al. (2015) found that students could suggest different ways to reach a solution, and certain students could even compare the different approaches and comment on which one was easier or harder. These studies (Chung & Tam, 2005; Göransson et al., 2015) concluded that teachers should change their "practice and drill" instruction methods and provide more instruction based on understanding mathematical concepts to their ID students. What remains to be answered is how capable teachers are when asked to provide this kind of instruction.

2.3 The Professional Knowledge of Mathematics Teachers

In the domain of mathematics, teachers must have undoubtedly acquired a deep understanding of the subject content and attempted to promote the active study of mathematics among their students. However, Dörfler and McLone (1986) noted that, to a large extent, a teacher's view of mathematics is shaped by their own experiences of learning mathematics as a student. Considerable curriculum changes that have occurred in recent decades mean that teachers are now required to teach a variety of mathematical content in ways that they have never experienced themselves. Consequently, professional development programmes must be organised to address the subject-specific concerns of teachers to improve the quality of mathematics teaching.

Mathematics teachers' knowledge can be divided into three strands: knowing mathematics, knowing teaching, and knowing how to teach mathematics (Liljedahl et al., 2009). According to Shulman's (1987) categories of teacher knowledge, the three strands belong to content knowledge (CK), general pedagogical knowledge (GPK), and pedagogical content knowledge (PCK). In the context of mathematics education, CK includes mathematical concepts, structures of mathematical knowledge, mathematical reasoning, mathematical thinking, and mathematical proofing. GPK is independent of individual subjects and "deal[s] with general principles of education such as theories of learning, sociological, psychological, and ethical aspects of education and its functions" (Liljedahl et al., 2009, p. 25). Last, Shulman (1987) describes PCK as follows:

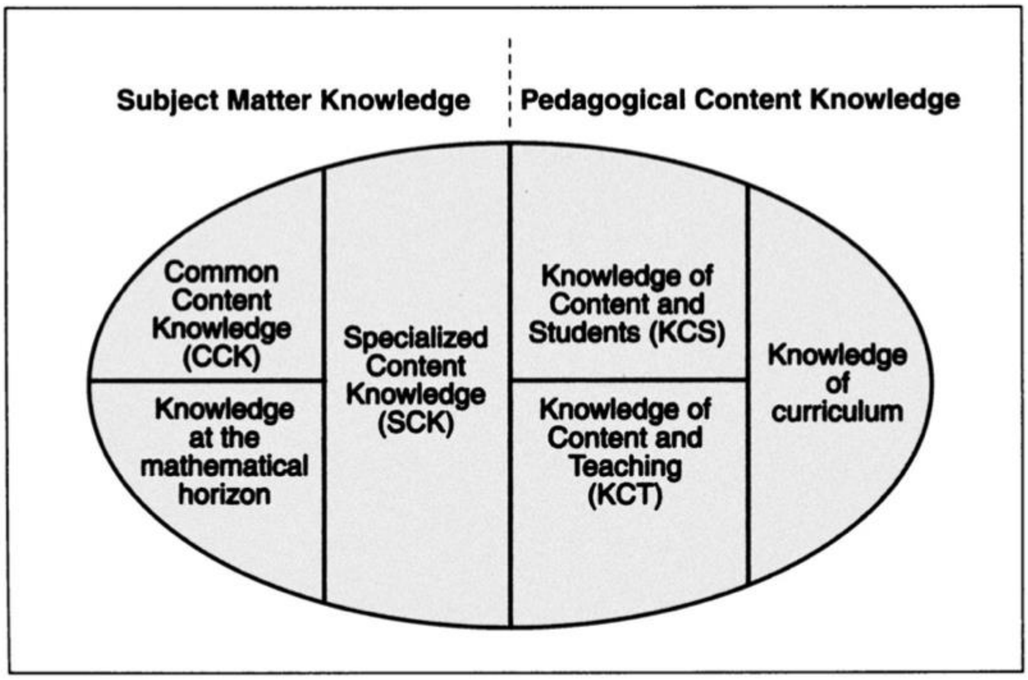
> "For the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others" (1986, p. 9)

Shulman was especially interested in PCK and believed it to be the category "most likely to distinguish the understanding of [a] content specialist from that of the pedagogue" (1987, p. 8). He also warned that the domain of PCK has long been a missing paradigm in teacher education (Shulman, 1986). Following Shulman's work, a large and growing body of literature has investigated knowledge models of mathematics teaching (Ball et al., 2008; Ernest, 1989; Fennema & Franke, 1992). Building on Shulman's (1986) framework of teachers' knowledge, Ball and her colleagues (Ball & Bass, 2009; Ball et al., 2008; Hill, Ball, et al., 2008; Hill, Blunk, et al., 2008) continued the investigation of mathematics knowledge for teaching. In their framework, PCK and specialised content knowledge (SCK) in mathematics are separate components of mathematical knowledge for teaching.

Figure 2.1 illustrates the mathematical knowledge framework for teaching developed by Ball and her colleagues (Ball et al., 2008; Hill, Ball, et al., 2008).

Figure 2.1

Domain Map of Mathematical Knowledge for Teaching



Note. From "Unpacking pedagogical content knowledge: Conceptualising and measuring teachers' topic-specific knowledge of students", by H. C. Hill, D. L. Ball, et al., 2008, *Journal for Research in Mathematics Education*, *39*(4), p. 377.

According to the above framework, PCK includes knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum. The domain of subject matter knowledge (SMK) also has three components. Among them, common content knowledge (CCK) represents the knowledge and skills used in a wide variety of settings that are not unique to teaching, for example, correctly solving mathematics problems. Specialised content knowledge (SCK) is mathematical knowledge that is not typically needed for purposes other than teaching, involving an uncanny unpacking of mathematics that is not needed in nonteaching settings.

Aside from the knowledge domains related to subject content, another essential knowledge needed in teaching is GPK, which includes principles and strategies of classroom management and organisation that are independent of individual subjects

(Shulman, 1987). Certain researchers have identified GPK as containing two components. One is pedagogical, which includes knowledge of teaching methods, classroom assessments, lesson structures, and adaptivity in dealing with heterogeneous learning groups in the classroom. The other component is psychological, which contains knowledge of various cognitive and motivational learning theories, learning strategies, and knowledge about individual student characteristics (Guerriero, 2014; König et al., 2011; Voss et al., 2011).

Some researchers have investigated how teachers' knowledge influences student learning outcomes. Hill (2005) found that teachers' knowledge of mathematical teaching positively relates to student achievement even in the instruction of very elementary mathematics content (Hill et al., 2005). Likewise, Baumert et al. (2010) found that compared to CK, PCK has a greater impact on student learning achievement and is decisive in determining the quality of instruction. However, research on the impact of teacher knowledge on student learning outcomes is scarce. The field thus needs more research to fully support the relationships between various domains of teacher knowledge and student learning outcomes (Guerriero, 2014). On the other hand, concerning the knowledge that teachers need to teach mathematics to ID students, Greer and Meyen (2009) suggested that teachers in special education classrooms need to improve their CK to translate curriculum standards into instructional methods that align with the needs of SEN students. Across several studies, teachers in special schools have been always found to have insufficient mathematical knowledge. Graham et al. (2000) suggested that the reason for this may be that the existing training programmes for special education teachers have been focused largely on instructional methods (GPK) rather than mathematical content (CK and PCK). In their project conducted to assist teachers in aligning their instructional methods with the needs of all students, Greer and Meyen (2009) found that the lessons teachers developed during the first few months were direct instructional methods that simply teach students to complete mathematics tasks step by step. Their experience suggests that teachers in special schools face a pressing need to raise their CK and PCK in mathematics teaching.

Considering the student characteristics and teaching environments of special schools, Brownell et al. (2009) stated that teachers working in special education also need subject knowledge targeting one-to-one instruction, as well as intervention adaptations for students' special needs, both of which are examples of PCK for special education. Darling-Hammond (1998) proposed that a deeper understanding of CK is the foundation for building PCK for special education. Teachers need to have a profound understanding of fundamental mathematics (Ma, 1999), as this understanding provides a foundation for organising and tailoring the learning content and making ideas accessible to students with disabilities (Darling-Hammond, 1998). PK is also necessary for teaching mathematics to ID students. As ID students may have difficulty expressing themselves, teachers need to be knowledgeable in interpreting their learning statements, understanding their strengths and weaknesses in learning to tailor suitable learning experiences for them, finding curriculum resources and technologies, and collaborating with other teachers or parents (Darling-Hammond, 1998).

Although researchers have made recommendations about the knowledge base teachers need for teaching SEN students, previous studies have failed to address the question of how to develop those kinds of knowledge.

2.4 The Training of Mathematics Teachers

Teachers play a vital role in the students' learning process. What teachers know and do is a product of their learning and teaching experiences. This section discusses a set of potential issues relevant to understanding the practices and programmes of in-service mathematics teacher education from an international perspective. It also describes the problems of traditional teacher education and how different countries and professional development programmes have been addressing these problems in the past decade.

Teacher training, regarding the teacher as a learner, aims to improve the quality of classroom teaching by equipping teachers with more professional knowledge. However, many teacher training activities have been criticised for not influencing teachers' practices when they returned to their classrooms (Parsad et al., 2001). One reason is that training activities such as workshops, conferences, and lectures are time limited. There is insufficient time for teachers to go into the content in-depth, and such training activities seldom provide follow-up and continuous support for teachers to apply the new knowledge to their practice (Garet et al., 2001). Teacher training activities are also ineffective if they fail to address teachers' learning needs. In mathematics education, teachers deal with various methods, theories, objects, and expectations from different educational bodies, yet what they need most is well-substantiated instructional designs and teaching resources that can be tested and modified in their own practice (Gravemeijer & van Eerde, 2009). Having collected similar comments from a group of teachers for SEN students, Kimmel et al. (1999) also observed that despite understanding the need for tailoring practices to cater to SEN students, teachers still suffer from various difficulties in administering adaptations in their classroom of students with differing learning needs. In sum, professional training is isolated from the teachers' classroom contexts and does not support them in overcoming their practical challenges.

The lack of effective professional development programmes has caused shifts in both their approaches and contents. For instance, Wittmann (1984) elaborated on the notion of the "philosophy of teaching units". He suggested that a teaching unit is not a detailed instructional planning of a series of lessons but "an idea for a teaching approach that leaves various open ways of realising the unit". He believed that "most teacher training programmes consist of isolated mathematical, education, didactical and practical components", but research in mathematics education most often "lacks the interlocking of different aspects" (Wittmann, 1984, p. 28). This concept of teaching units provides a potential way of integrating all the components of teacher education into mathematics teaching practice. Using teaching units, teachers can test the instructional designs they developed in their practice, in which they become contributors who display initiative in modifying and further developing the designs into more localised versions appropriate for their classroom contexts, instead of remaining passive consumers of generic designs (Gravemeijer & van Eerde, 2009). This notion of teaching units is considered the core of "mathematics education as design science" (Wittmann, 1984, 1995, 2001).

Last, collaboration between experts in different fields related to education has been regarded as one of the most effective approaches in the educational planning process (Hunt et al., 2003). Each collaborative team member has a unique set of expertise; through the collaborative process, shared understandings of individual knowledge can be achieved (Horn & Kang, 2012). Design research in education, such as the design and refinement of teaching units, proposes a close collaboration between researchers and teachers. This partnership benefits both groups, as it fosters an exchange of ideas and findings, which "makes the research more practical and the teaching more scientific" (Gravemeijer & van Eerde, 2009, p. 511). Through conversations and learning processes, the gap between researchers and teachers is narrowed.

The collaborative team model has also been recommended for designing an educational process for special education learners (Horn & Kang, 2012). Given that in-service teachers know their students the best, they are best able to design activities that address their students' learning needs. However, teachers may lack adequate subject content knowledge to ensure strong subject relevance and richness in learning activities. Subject content advice from experts is therefore needed for analysing and planning learning activities. Collaboration between subject experts and teachers is perceived to have a significant impact on the outcomes for both students and team members alike. Many research studies have discussed the effectiveness of collaboration in the instructional design process, but far less information exists about to what extent the collaborative model works in promoting professional development among special education teachers. As participants and advisors in the BE MATHS programme collaborate closely throughout the instructional design and testing process, the present study can evaluate the effects of collaboration on the programme.

2.5 A Conceptual Framework for Professional Development Programmes

The professional development of teachers is an ongoing and continuous learning process in which teachers experience a wide range of activities and interactions that aim to enhance their skills and knowledge in teaching (Desimone, 2009). The experiences can be either formal or informal and range from structured training programmes to casual conversations with other teachers about instructional methods. As learning opportunities for professional development are embedded in teachers' daily lives, it is difficult to distinguish learning activities from each other when measuring the effectiveness of a professional development intervention. Desimone (2009) suggested a solution to this challenge:

> "One way of translating the complex, interactive, formal, and informal nature of teacher learning opportunities into manageable, measurable phenomena is to focus measurement on the critical features of the activity—those characteristics of an activity that make it effective for increasing teacher learning and changing practice, and ultimately for improving student learning—rather than on the type of activity (e.g., workshop or study group)" (p. 183).

Researchers have since reached a consensus on the core features of high-quality professional development (Garet et al., 2001; Hawley & Valli, 1999). These features include (1) content focus—activities that focus on subject content as well as on how students learn that content; (2) active learning—as opposed to passive learning, such as listening to a lecture; (3) coherence—consistency between different activities; (4) duration—providing enough time for professional development activities; and (5) collective participation—activities designed for teachers in the same school or teaching at the same student level (Desimone, 2009; Firestone et al., 2005).

Among the five features, content focus is the most influential. Research on teacher learning has widely shown that activities or teaching materials that focus on subject content knowledge and PCK improve teachers' practice and students' achievement. For example, Luft et al. (2011) found that science teachers enrolled in a science-specific mentoring programme performed better than teachers enrolled in a general mentoring programme. In a case study describing how a secondary biology teacher mentor influenced a preservice teacher mentee's professional development, Barrouillet and Lépine (2005) found that discussions between mentor and mentee always focused on a concrete topic through which the mentee developed science-specific PCK. Last, through their systematic review of intervention studies aimed at PCK development, Evens et al. (2015) found that effective interventions always included learning activities on both student understanding and representations of subject matter. In contrast, interventions that addressed knowledge in classroom management were less effective or not effective at all.

Regarding the second feature, namely, active learning, teachers are directly involved in solving teaching problems, designing teaching materials, and practising new strategies. In the professional development programme implemented under the study by Girvan et al. (2016), an experiential learning approach was used. Participating teachers experienced new instructional strategies themselves as learners before tailoring the strategies to their students and implementing them in practice. The findings by Girvan et al. (2016) suggested that experiential learning activities helped teachers move from teacher-centred to learner-centred approaches. In addition, their students were also observed to be more actively engaged and more confident in their learning. Other forms of active learning include participating in lesson observations, reviewing students' learning materials, and discussing teaching problems with other teachers.

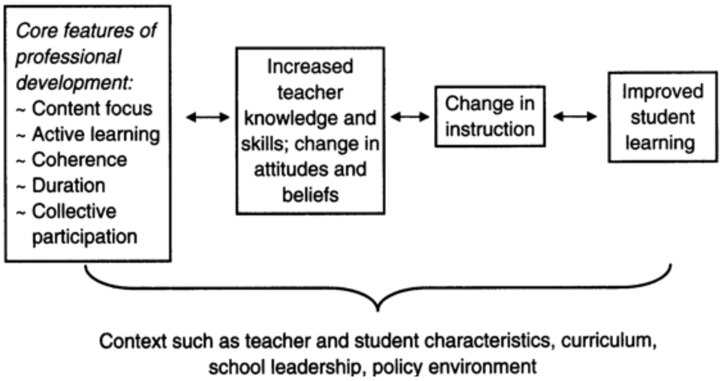
The third feature, namely, coherence, is often recommended for professional development but seldom defined clearly (Lindvall & Ryve, 2019). Firestone et al. (2005) considered a professional development activity to be coherent when it addresses less content in more depth and has more adequate follow-up. A teacher training activity should develop in-depth knowledge to support teachers facing extensive changes and addressing challenges in practice. On the other hand, an activity that covers too many topics will leave teacher participants with only a superficial understanding of its content. Another element of coherence is the provision of continuing support for teachers' professional development. Teachers prefer long-term training, as it gives them opportunities to try new teaching ideas, reflect on them, and refine them with guidance from teacher educators (Firestone et al., 2005).

The fourth feature, namely, duration, refers to "the span of time over which the activity is spread, and the number of hours spent in the activity" (Desimone, 2009, p. 184). Past studies have shown that the more time teachers commit to professional development activities, the more likely they are to improve their teaching practices (Garet et al., 2001).

Last, the fifth feature, namely, collective participation, in a professional development programme that promotes collaboration among teachers from the same school or teachers who have students at similar achievement levels. Through collaboration, the teachers can share teaching materials and curricula and easily understand each other's teaching problems.

Based on the five core features, Desimone (2009) constructed a conceptual framework for studying teacher professional development (see Figure 2.2). To determine whether an intervention on teaching quality is successful, teachers' knowledge, skills, beliefs, and attitudes should all be considered. In addition, student performance serves as an important indicator for evaluating the outcomes of a professional development programme (Hartman, 2010).

Figure 2.2
Proposed Framework for Studying the Effects of Professional Development on Teachers and Students



Note. From "Improving impact studies of teachers' professional development: Towards better conceptualisations and measures", by L. M. Desimone, 2009, *Educational Researcher*, *38*(3), p. 185.

Concerning professional development that is specific to mathematics teaching for ID students, the research base in terms of the number of studies conducted on relevant programmes remains limited. Even when they expanded the target student group from only ID students to students with learning difficulties in mathematics in their meta-analysis review, Allsopp and Haley (2015) found only 16 relevant studies in the last decade, of which ten discussed a professional development intervention and nine examined the effects of intervention on preservice or in-service teachers. Due to the small number and varied nature of the 16 studies, Allsopp and Haley (2015) found it difficult to reach any conclusions. There is thus a lack of evidence for identifying critical features that constitute effective professional development in the specific context of mathematics teaching for ID students. Allsopp and Haley (2015) also reported that both preservice and in-service teachers, in general education and special education alike, have low self-efficacy for teaching mathematics, poor learning experiences in mathematics, and high levels of anxiety. However, no studies have explored how teachers' personal attributes and experiences influence their teaching and their students' achievements. The authors therefore suggested that future research connect teacher scores (e.g., self-efficacy, mathematics anxiety, mathematical knowledge) to their actual practice or student outcomes. Last, Allsopp and Haley (2015) strongly suggested that researchers should include student outcomes in studies on professional development interventions. Among the 16 studies they found, only two mentioned the impact of interventions on students' achievements. Considering Allsopp and Haley's findings and suggestions, to provide a comprehensive picture of the effect of the BE MATHS programme on teachers' professional development, the current study should consider whether ID students' performance has improved during the programme period. Section 2.7 will discuss the difficulties in evaluating intervention effectiveness in ID students and suggest "measuring academic engaged time" as a workaround to bypass such difficulties.

2.6 Mathematics Teaching Efficacy Beliefs

The previous section mentioned findings by Allsopp and Haley (2015) that reported teachers experiencing low self-efficacy in regard to teaching mathematics. Self-efficacy refers to the belief "in one's capacity to organise and execute the courses of action required to produce given attainments" (Bandura, 1997, p.3). Teaching efficacy belief is one type of self-efficacy that refers to how much a teacher believes that they can accomplish mathematics teaching and influence students' mathematics learning outcomes. This belief influences teachers' willingness to experiment with instructional ideas (Bruce & Ross, 2008). Teachers with high teaching efficacy use effective classroom teaching strategies to encourage student autonomy, meet the needs of low-ability students, and positively influence student perceptions of their abilities (Ross, 1998).

Mathematics teaching efficacy beliefs describe the beliefs that a teacher holds about their ability to accomplish mathematics teaching. Previous research has shown these beliefs to be a key factor in teacher development and a potent influence on teaching practices and student achievement. One particular component of teaching efficacy is outcome expectancies, which refers to teachers' beliefs about the effects that their teaching actions will have on students. A teacher may be highly confident in their ability to execute a teaching task but also doubt the outcome. High outcome expectancy reflects the degree to which a teacher or a group of teachers believe that teaching actions can control teaching outcomes. In several studies aimed at enhancing teacher efficacy beliefs, researchers have found that teachers' growth in mathematics teaching efficacy can result from receiving positive feedback from their peer coaching partners, acquiring and applying new instructional strategies in their classrooms, and gaining mastery experiences (Bruce & Ross, 2008).

Enochs et al. (2000) developed the Mathematics Teaching Efficacy Belief Instrument (MTEBI) to measure such beliefs. The instrument comprises two subscales, one measuring personal mathematics teaching efficacy (PMTE) and the other measuring mathematics teaching outcome expectancy (MTOE). PMTE refers to how much confidence a teacher has in their mathematics teaching abilities, while MTOE shows how much a teacher believes that effective teaching can influence students' mathematics learning outcomes (Enochs et al., 2000). Various studies have since used the instrument to measure the teaching efficacy beliefs of both preservice (Aksu & Kul, 2019; Dofková, 2007; Moody & DuCloux, 2015; Swars et al., 2006; Swars et al., 2018; Wenner, 2001) and in-service mathematics teachers (Özben & Kilicoglu, 2021; Wenner, 2001).

2.7 Measuring the Academic Engaged Time of ID Students

The collection of data on ID students' learning performance aims to discover evidence that can reflect increases or decreases in teachers' teaching capabilities. Collecting or analysing performance data on ID students is difficult for the following reasons:

1. ID students belong to a low-incidence population (Spooner & Browder, 2003), which always results in a small sample size study and prevents researchers from obtaining statistical significance (Browder et al., 2008; Mertens, 2014);
2. ID students usually have more than one significant disability, such as autism, sensory impairments, or behavioural disorders (Ageranioti-Bélanger et al., 2012; Kiani & Miller, 2010; Vaan, 2013), which makes the ID student population very heterogeneous. The diversity regarding types and degrees of disability significantly reduces the internal validity of any comparison of educational outcomes between different groups of ID students; and
3. ID students are likely to have a delay in language development. Hence, relying on communication responses to indicate their learning statements makes it

substantially challenging for researchers to code data reliably (Spooner & Browder, 2015).

In the literature, data collection exercises on ID students' mathematics learning typically adopt single-subject research designs to evaluate whether teaching interventions affect their learning. The control strategies in single-subject research designs (e.g., the A-B-A and A-B-A-B approaches), which involve showing a strong reversal from baseline to treatment and back again, make the comparison of students' performance with and without intervention feasible. However, the current study cannot apply this approach because the BE MATHS programme's intervention on teachers' professional knowledge is irreversible. Moreover, it may not be ethical for both the researchers and the teacher participants to stop their instruction once they see evidence of its effectiveness on ID students.

Measuring ID students' academic engaged time is a potential workaround to bypass the difficulties in evaluating students' learning achievements directly. Academic engaged time, also known as "time-on-task", refers to instances that show evidence of learning engagement. For example, a student may answer their teacher's question or react to their teacher's instructional actions (Johns et al., 2008). Student engagement has been described as a composite of specific responses to instruction, including actions such as reading aloud and asking or answering questions. In the past decade, research has found that student engagement is sensitive to changes in instruction, including teacher behaviours, learning materials, or instructional methodologies (Greenwood, 1991). Moreover, studies have also shown that students' engagement in learning can predict their academic achievement. Academic engaged time and engagement rate are all positively associated with student achievement. Students who accumulate more academic engaged time generally have higher scores on achievement tests (Fisher et al., 1981; Greenwood et al., 2002). Considering the characteristics of ID students, measuring their academic engaged time could help researchers avoid unreliable interpretations of their learning progress. Taking such measurements can also provide quantitative data for statistical analysis, which can improve the reliability of results and make the statistical control of students' heterogeneous backgrounds possible.

Chapter 3 Research Questions and Methods

3.1 Research Questions

As mentioned in Chapter 1, subject-specific support in mathematics teaching for ID students has been little studied in the academic field and is uncommon in the practice of teacher education. Given that teachers face significant challenges in teaching mathematics to ID students, it is necessary to explore this type of support. To explore whether and how subject-specific support can contribute to teachers' profession in teaching mathematics to ID students, the current study examines the effects of the BE MATHS programme and the contributing factors of the effects.

The study focuses on the following four major research questions (RQs):

1. How do the two groups of teachers (BE MATHS group and control group) view the characteristics of their professional development experience in terms of active learning inside and outside the classroom, coherence with their needs, collective participation, and content focuses?
2. What changes do teachers exhibit in teaching mathematics to ID students after participating in the BE MATHS programme?
3. What changes do students exhibit in their engagement with mathematics learning after their teachers participated in the BE MATHS programme?
4. How does the BE MATHS programme influence teachers' mathematics teaching?

RQ1 examines the characteristics of teachers' professional development experiences in the programme. The answers to RQ2 and RQ3 report the effects of the programme on teachers' mathematics teaching efficacy and students' academic engaged time in mathematics classes. Last, RQ4 tries to understand the programme and its contributing factors using qualitative approaches.

3.2 Research Design

The study employed a mixed-methods approach to answer the research questions. It collected and analysed both quantitative and qualitative data to understand the professional development of teachers in the BE MATHS programme. Given the dynamic nature of teachers' professional development (Desimone, 2009), the combination of quantitative and qualitative enquiries provides a more comprehensive understanding of teacher learning than either approach could offer on its own (Creswell & Plano Clark, 2018). In this research, quantitative data were collected to measure teachers' mathematics teaching efficacy and students' academic engaged time in mathematics classes. Quantitative data analysis assesses the programme outcomes in terms of changes to teachers and students. To gain a more in-depth understanding of how the programme actually contributes to those changes, qualitative data were also collected. By assessing both the outcomes and the process of the programme, the study can develop a rich and comprehensive picture of professional development in mathematics teaching for ID students.

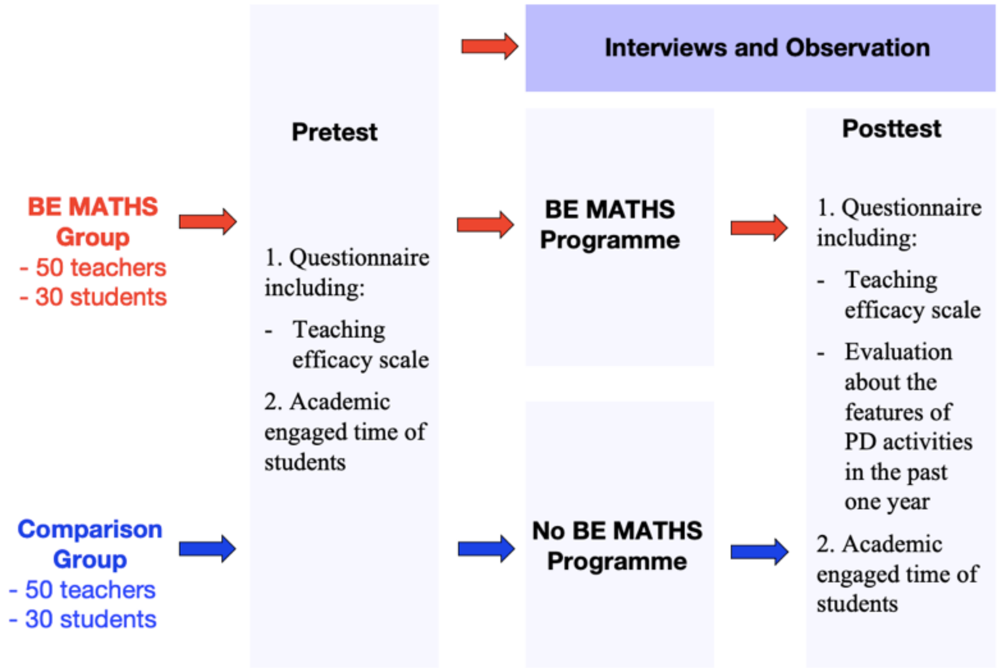
The study collected quantitative data on teachers' mathematics teaching efficacy beliefs, teachers' evaluations of the five professional development features of the BE MATHS programme, and ID students' academic engaged time during the mathematics classes taught by participating teachers in the programme. Three instruments and an online survey were employed to measure one independent variable, seven dependent variables, and demographic control variables. Table 3.1 displays the variables and the associated instruments used to measure them.

Table 3.1
Variables and Instruments for This Study

Variable Type	Variables	Instruments
Independent variable	Participation in the BE MATHS programme	Online survey
Dependent variables	Mathematics teaching efficacy scores	Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) scale (Enochs et al., 2000)
	Student academic engaged time	Individual Child Engagement Record-Revised (ICER-R) scale (Kishida & Kemp, 2006; Kishida et al., 2008)
	Score for active learning inside classroom	Characteristics of Teacher Professional Development (CTPD) scales (Soine & Lumpe, 2014)
	Score for active learning outside classroom	
	Score for content focus	
	Score for coherence with needs	
	Score for collective participation	
Control (demographic) variables	Years of teaching in special schools	Online survey
	Students' ID level	
	Teacher's mathematics education background	

Qualitative data were collected through interviews and observations in the professional development programme to gain in-depth insights into the programme. Data were collected on teachers' experiences with the programme, including teaching skills and their students' performance. The data on teachers' narratives and experiences were expected to help form contextualised links to the quantitative outcomes of this study. Figure 3.1 summarises the study process.

Figure 3.1
Research Design



The research design comprised two components. The first component aimed to understand the effect of the BE MATHS programme on teachers' teaching efficacy and their students' academic engaged time in mathematics classes (RQ2 and RQ3). Participants in the BE MATHS programme were invited to participate in this study as members of the intervention group, while teachers who had not participated in the programme were invited to participate as members of the control group. A survey including the MTEBI scale and a background information questionnaire was developed to collect data on the independent and control variables listed in Table 3.1.

This component of the study also aimed to investigate whether a difference exists between control and intervention group teachers in terms of teaching efficacy and students' academic engaged time. As both variables are influenced by many factors (e.g., teachers' educational background, teachers' prior teaching experience, students' prior knowledge, and students' degrees of disability), the study used regression techniques to control the observable differences between the groups of teachers and students.

The second component aimed to explore teachers' professional development experiences in the BE MATHS programme and identify the programme factors that improve their teaching for ID students (RQ4). The Characteristics of Teacher Professional Development (CTPD) scale (Soine & Lumpe, 2014) was used to measure teachers' perceptions about the characteristics of their professional development both inside and outside the programme. In addition, interviews and observations were conducted to explore teachers' experiences of teaching mathematics to ID students and of professional development support during the BE MATHS programme. All the quantitative and qualitative data collected through the above methods were first analysed separately and then triangulated for interpretation and validation.

3.3 Participants and Recruitment Procedures

Approximately 50 special school mathematics teachers participated in the BE MATHS programme during the 2017–18 school year. All teachers in that programme cohort were invited to participate in this study and organised into several intervention groups. Another group of teachers was recruited to serve as a control group and provide references for the comparison of teaching efficacy scores between teachers participating in the intervention and teachers not included in the intervention.

Participation in this research was voluntary. An invitation email about the research was sent to all participants in the BE MATHS programme to explain the purpose of the study and seek their cooperation. To invite control group participants, another invitation email was sent to the administrators of all Hong Kong special schools with ID students to ask for assistance in recruiting mathematics teachers. After receiving approval from both the principals and the teachers, a link to a survey formulated by the researcher was forwarded by email to the teachers. The email acknowledged that participation in the study was voluntary and invited teachers to complete the survey through the link provided if they were willing to.

The study observed students' performance in the mathematics classes of participating teachers. Currently, ID students in Hong Kong are categorised into three groups: mild (IQ from 50 to 69), moderate (IQ from 25 to 49), and severe (IQ below 24). Of the ten schools that participated in the BE MATHS programme during the 2017–18 school year, two for mild ID students, two for moderate ID students, and two for severe ID students were randomly selected for participant recruitment. Among those schools that did not participate in the BE MATHS programme, two for mild ID students, two for moderate ID students, and two for severe ID students were also selected through referral. After informing the schools' principals and teachers about the purpose of the study, the researcher asked for their assistance in contacting their students' parents to obtain consent regarding the students' participation in the study. Teachers delivered consent forms to the parents, accompanied by a cover letter stating the research purpose and content so that they had adequate information to decide whether they would allow their child to join the study.

3.4 Instruments

3.4.1 Characteristics of Teacher Professional Development (CTPD)

The five core professional development features mentioned in Chapter 2—content focus, active learning, coherence duration, and collective participation—were measured using the Characteristics of Teacher Professional Development (CTPD) scale designed by Soine and Lumpe (2014). The scale was developed following an exploratory factor analysis of pilot data on teachers' perceptions of characteristics of professional development in their study, in which Soine and Lumpe (2014) found that certain items did not group together the way they had predicted. First, the items intending to measure active learning in professional development programmes fell into two different categories, namely, active learning in the classroom and beyond the classroom. Moreover, one item ("spread evenly throughout the school year") intended to measure duration was found to load strongly on the component of coherence. Soine and Lumpe (2014) therefore clustered this item under coherence and eliminated other items intended to measure duration. Based on the components that emerged from the factor analysis results, the improved version of their instrument includes the five new subscales of (1) collective participation, (2) focus on teachers' CK and how students learn content, (3) coherence with teachers' needs and circumstances, (4) active learning in the classroom, and (5) active learning outside the classroom. All the new subscales reported acceptable Cronbach's alpha scores ($\alpha > .70$), as listed in Table 3.2.

Table 3.2
Cronbach's Alpha Scores for CTPD Subscales

CTPD Subscale	Number of items	Cronbach's alpha
Collective participation	16	.91
Focus on teachers' CK and how students learn content	14	.94
Coherence with teachers' needs and circumstances	9	.90
Active learning in the classroom	7	.81
Active learning outside the classroom	6	.72

Note. Cronbach's alpha scores for each subscale after factor analysis reported in "Measuring characteristics of teacher professional development", by K. Soine & A. Lumpe, 2014, *Teacher Development*, *18*(3), p. 317.

To measure the participants' perceptions about the features of the BE MATHS programme, a Chinese translation of the CTPD was needed. The translation procedure was the same as that used for translating the MTEBI scale (Enochs et al., 2000) and is detailed in the following section.

3.4.2 Mathematics Teaching Efficacy Beliefs Instrument Scale (Chinese Version)

The study employed a translated Chinese version of the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) scale (Enochs et al., 2000) to measure the effects of the BE MATHS programme on mathematics teaching efficacy. The original MTEBI scale comprises two subscales with a total of 21 items. The Personal Mathematics Teaching Efficacy (PMTE) subscale consists of 13 items, and the Mathematics Teaching Outcome Expectancy (MTOE) subscale consists of 8 items. Each item uses a 5-point Likert scale to let respondents indicate how much they agree with the statements of each item, ranging from "strongly disagree" to "strongly agree". Enochs et al. (2000) reported the result of reliability analysis that the scale produced an alpha coefficient of α=.88.

The current study needed to translate and adapt the scale into a version that was appropriate for the local Hong Kong in-service mathematics teachers in this study. The translation of instruments is not a simple word-for-word conversion process, as it needs to consider the cultural and linguistic differences that distinguish the target-language subjects. Following the guidelines proposed by Sousa and Rojjanasrirat (2011), who have reviewed available and highly recommended approaches to the translation and cross-cultural validation of research instruments, this study employed the following steps to strengthen instrument validity:

1. *Forward translation.* The original instrument was delivered to two independent translators for translation into Chinese. Both translators spoke Cantonese as their mother tongue and were fluent in English. Moreover, one of the translators was knowledgeable about mathematics teaching and translated expressions in the scale to ensure that they were familiar to the local mathematics teachers.
2. *Synthesis I*. A committee comprising the two translators in step 1 and the project investigator compared the two forward-translated versions of the instrument and discussed discrepancies in words, sentences, and meanings. Consensus was achieved during the meeting, and a preliminary translated version of the instrument was generated.
3. *Blind back-translation.* The preliminarily translated instrument was translated back into English by two additional independent translators who were completely blind to the original instrument. One of the translators is a bilingual person who is a fluent native speaker of both Chinese and English. He has a Bachelor of Education degree, with a major in English. The other translator is an English teacher whose first language is Chinese.

4. *Synthesis II.* A multidisciplinary committee compared the two back-translations of the instrument with the original version to evaluate the similarity of the instructions, in addition to items and response formats with respect to wording, sentence structure, meaning, and relevance. The committee comprised the project investigator, a professional in the field of efficacy, and all four translators involved in steps 1 and 3. Discrepancies were resolved at this stage; in the case that they were not, the investigator needed to recruit other translators to repeat steps 1 through 3, thereby ensuring that the retranslated and back-translated items retained the original meanings of the instrument.
5. *Cognitive interviewing*. To improve the conceptual and content equivalence of the instrument items, this study conducted cognitive interviews with five mathematics teachers from special schools. Five teachers were recruited using convenience sampling and did not participate in the primary study. After they completed the instrument, the teachers were probed further about their responses in the interviews. For instance, the interviewer asked the teachers to paraphrase the items to see whether they reached an understanding similar to the intended meaning of the items. Based on the results and suggested revisions gained from the cognitive interviews, the instrument was further modified into a final version.

The translated Chinese MTEBI scale was incorporated into an online survey questionnaire used to evaluate the effects of the BE MATHS programme on teachers' mathematics teaching efficacy. Mathematics teachers from 41 ID schools in Hong Kong were invited to complete the survey. Pretest and posttest analyses were enabled by administering the questionnaire at both the beginning and the end of the school year.

All the items in both iterations of the questionnaire were identical except for certain demographic questions that were deleted from the posttest iteration. Concerning the potential non-equivalence between the teachers in and outside of the BE MATHS programme, such as years of teaching in special schools, class sizes, and educational backgrounds, the study used regression techniques to control factors that may influence the scores indicating the effects of the programme on teachers' efficacy.

3.4.3 Observations

A series of observations were conducted on professional development activities that occurred during the BE MATHS programme, such as collaborative planning in learning circle meetings and teaching experiments. During observation, the observer was interested in (i) what problems teachers faced regarding teaching mathematics to ID students and how the programme addressed these problems, in addition to (ii) how the programme developed teachers' knowledge of mathematics and its teaching. Following Creswell and Plano Clark (2018) suggestions that researchers use pre-designed protocols to guide and organise their thoughts during observations, observation protocols (Appendices 1 and 2) were designed for the study and used by observers during observations.

Another type of observation activity conducted in this study consisted of lesson observations that examined student performance. Given that such observations are very time-consuming, it was unfeasible to observe all the students of all the participating teachers in this study. The study instead proposed observing 30 ID students each from the intervention and control groups. Each student was observed twice. During lesson observations, the researcher attempted to discover (1) the performance of ID students, (2) the observable progress that ID students made, and (3) how long the ID students were engaged in learning. The observation findings were used as evidence for evaluating the effect of the BE MATHS programme on teachers' practice and students' performance.

In addition, student performance was evaluated by measuring student's academic engaged time in a mathematics class. In this study, academic engaged time was defined as the time in which the target student was appropriately engaged in a mathematics learning activity. Specifically, academic engaged time was counted when the target student was listening to a teacher, responding appropriately to instruction (e.g., following directions, manipulating materials), pointing appropriately to learning materials, asking questions, showing their work to a peer, or responding to a teacher's questions. Invalid examples of academic engaged time included but were not limited to when the target student was not looking at the teacher, was running in the classroom, was inappropriately using materials, or was singing songs that were unrelated to the learning contents. A 15-second momentary time-sampling procedure was used to observe individual student engagement. First, the student's behaviour in the mathematics class was recorded on video. The video recordings were then split into shorter 15-second video clips that each served as an observation interval. Observers watched each interval and recorded whether the student was engaged in mathematics learning at the very end. The number of instances of engagement recorded per mathematics class was counted and then calculated as a percentage of the intervals to compare the results from different classes. The study employed the revised version of the Individual Child Engagement Record (ICER-R) instrument (Kishida & Kemp, 2006; Kishida et al., 2008) to measure different variables of academic engaged time.

For both types of observation activities, the researcher acted as a complete observer and quietly took notes most of the time, even though their presence was known and recognised by the subjects.

3.4.4 Teacher Interviews

The study interviewed some participating teachers of the BE MATHS programme to understand their perspectives about what changes they experienced in teaching mathematics to ID students, as well as how the programme influenced their mathematics teaching practice and students' performance. The interview content focused on teachers' learning experiences, their changes, and their students' changes in the BE MATHS programme. To gain a range of perspectives, the teachers interviewed were selected from the three different groups of special schools, namely, schools for mild, moderate, and severe ID students. At least one teacher from each group of schools was interviewed. Appendix 3 presents the interview protocol designed to guide these interviews.

Analysis of the qualitative data collected through observations and interviews occurred in six stages. In the first stage, the researcher read interview transcripts and field notes to obtain an initial sense of the data. The software NVivo was used to organise and analyse the transcripts of the interviews and field notes of observations. During the second stage, open coding was conducted to identify characteristics in the data that were relevant to professional development programme features; changes in teachers' attitudes, knowledge, and skills in teaching mathematics; and changes in students' learning performance. Table 3.3 presents the open coding list developed during this stage. The third stage involved collapsing codes and defining examples and non-examples for each code. In the fourth stage, the researcher used the defined codes to code the transcripts and field notes stored in NVivo. In the fifth stage, the researcher linked the codes together and tried to explain the interrelations between them. In the last stage, the qualitative findings were triangulated with the quantitative findings to produce more objective and more complete interpretations of the data.

Table 3.3
List of Open Coding Codes for Qualitative Data

Category	Code
Knowledge	Content knowledge (CK)
	Pedagogical content knowledge (PCK)
	Pedagogical knowledge (PK)
Teaching methods	Direct teaching
	Teaching for conceptual understanding
Difficulties in teaching	Lack of materials
	Lack of knowledge
	Difficulty in catering to student's learning disabilities
Professional development features	Content focus
	Active learning
	Coherence
	Duration
	Collective participation

3.5. Ethical Considerations

Before the research began, the researcher applied for ethical review from the Human Research Ethics Committee. The current research was not seen as a violation of participant rights. Participating teachers and students could freely choose not to participate in the study with no repercussions. In particular, the cover letter distributed to students' parents clearly explained the research purpose and content to them so that they had adequate information to decide whether they would allow their child to join the study.

As the online survey questionnaire was delivered to teachers through their principals, the teachers may have felt pressured to complete it. The researcher thus requested that principals emphasise to the teachers that they were free to decline participation in the study with no consequences, which the survey email reaffirmed. Moreover, given that the teachers submitted their questionnaire responses online, the principals were unaware whether teachers decided to participate and what their answers to the questionnaire were, thereby preserving the teachers' anonymity. Another ethical risk was that the online survey collected teachers' names to compare their pretest and posttest scores. The researcher clearly acknowledged this issue in the survey email and at the start of the online questionnaire. To minimise privacy risks, when the researcher received questionnaire responses, any identifying information for individual teachers was kept secret using codes known only to the researcher. The rest of the subjects' data were also kept secret to ensure that their identities would not be revealed.

Neither the online survey nor the interviews covered any sensitive topics or attempted to invoke any painful memories. When the pilot study was completed, the questionnaire was checked for any information that might cause discomfort, embarrassment, or other negative emotional responses. Furthermore, all the participants were informed of their right to discontinue their participation in the study at any time for any reason.

The final ethical concern was that certain special school lesson observation sessions were video recorded, with certain ID students on those recordings being selected for further analysis regarding their performance. To obtain the right to create recordings of the students, the researcher asked participating teachers for their assistance in delivering consent forms to students' parents; thus, images of the students were captured only with parental permission. The video recordings were kept in a secure server and deleted after transcripts were made for them.

Chapter 4 Quantitative Findings

This chapter presents the quantitative findings generated from the questionnaires and student academic engaged time measured in mathematics classes to address the following research questions:

- RQ1: How do the two groups of teachers (BE MATHS group and control group) view the characteristics of their professional development experience in terms of active learning inside and outside the classroom, coherence with their needs, collective participation, and content focuses?
- RQ2: What changes do teachers exhibit in regard to teaching mathematics to ID students after participating in the BE MATHS programme?
- RQ3: What changes do students exhibit in their engagement with mathematics learning after their teachers participate in the BE MATHS programme?

4.1 Sample

4.1.1 Sample of Teachers

Teacher participants were recruited by email and sent to 49 special schools for ID students in Hong Kong. The teachers responded independently to three tests, namely a pretest of mathematics teaching efficacy beliefs, a posttest of mathematics teaching efficacy beliefs, and the CTPD test. As shown in the Venn diagram in Figure 4.1, 403 teachers completed the pretest, 267 teachers completed the posttest, and 125 teachers completed the CTPD test. Only the teachers who completed all three tests were considered in the analysis; teachers who did not complete all three tests were deleted from the dataset. The final sample size of teachers for this study was N=96. The sample included 32 teachers who participated in the BE MATHS programme (BE MATHS group) and 64 teachers who did not participate in the programme (control group); the teachers represented 27 special schools between them.

Figure 4.1
Venn Diagram for Samples of Teachers

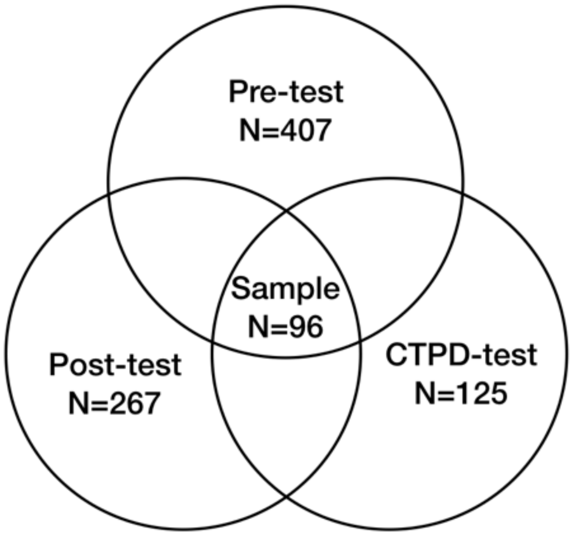


Table 4.1 summarises the demographic information of the two groups of teachers. Most teachers in both groups had taught students with mild and/or moderate ID, with the proportion of control group teachers who taught students with moderate ID being 14% higher than that in the BE MATHS group. Both groups also had similar proportions of teachers with over 10 years of teaching experience at special schools and teachers who had received no professional mathematics teacher training.

Table 4.1
Teacher Participant Demographics

	No. of teachers (%)	
Teacher characteristics	BE MATHS group (*N*=32)	Control group (*N*=64)
Teaching mild ID students	22 (68.8)	43 (67.2)
Teaching moderate ID students	18 (56.3)	45 (70.3)
Teaching severe ID students	8 (25.0)	13 (20.3)
>10 years of special school teaching experience	13 (40.6)	28 (43.8)
No mathematics training	10 (31.3)	19 (29.7)

4.1.2 Sample of Students

The study planned to videotape 20 students from each of the three groups of mild, moderate, and severe ID students. However, during the recruitment process, the researcher found that the more severe the ID level of a student, the greater the difficulty in gaining permission from the school to record them. Certain school principals rejected the request to record their students and explained that many of them were orphans that required additional protection to protect their identity. Some students' guardians also rejected the request due to ethical concerns, even though they understood that the researcher would earnestly protect their child's personal information. Thus, the final video study sample included 83 students, 55 from the BE MATHS group and 28 from the control group. In the BE MATHS group, 40% of students had mild ID, 40% had moderate ID, and 20% had severe ID. In the control group, more than half of the sample (57%) had mild ID, 28.7% had moderate ID, and 14.9% had severe ID. The proportion of mild ID students in the control group relative to the entire group was approximately 1.5 times greater than that in the BE MATHS group (see Table 4.2).

Table 4.2
Student Participant Demographics

	No. of students (%)	
ID level	BE MATHS group (*N*=55)	Control group (*N*=28)
Mild	22 (40.00)	16 (57.14)
Moderate	22 (40.00)	8 (28.57)
Severe	11 (20.00)	4 (14.29)

4.2 Professional Development Experience in the BE MATHS Programme

Teachers' evaluations of the five core features of their professional development experience were collected through their responses to the CTPD test. The variables of interest for data analysis are listed as follows:

- *ALIC*: a continuous variable that is a participant's score on the subscale measuring **active learning in the classroom** during the programme;
- *ALBC*: a continuous variable that is a participant's score on the subscale measuring **active learning beyond the classroom** during the programme;
- *CKT*: a continuous variable that is a participant's score on the subscale measuring the **focus on teachers' content knowledge** (CK) and how students learn content during the programme;
- *CWNC*: a continuous variable that is a participant's score on the subscale measuring the programme's **coherence with teachers' needs and circumstances**;
- *CP*: a continuous variable that is a participant's score on the subscale measuring **collective participation** in the programme; and
- *Participation*: a categorical variable that takes a value of 1 for teachers who participated in the BE MATHS programme and a value of 0 for teachers who

did not participate.

To understand in what aspects the BE MATHS programme activities differed from the professional development activities that teachers had experience participating in, an independent-sample *t*-test was conducted to compare the continuous variables measuring the five core features of teacher professional development activities (ALIC, ALBC, CKT, CWNC, and CP). Table 4.3 shows that the mean scores for all five features among the BE MATHS group teachers were significantly higher than those for the control group teachers ($p \leq .05$ or $p \leq .001$). These results suggest that compared to typical professional development activities, BE MATHS programme activities encourage more collective participation, are more content-focused, are more coherent with teachers' needs, and involve more active learning both inside and outside the classroom.

Table 4.3
Descriptive Statistics and Independent-Sample t-Test Results for PD Experience Evaluation Scores

	BE MATHS group (*N*=32)		Control group (*N*=64)		Test results	
Variable	*M*	*SD*	*M*	*SD*	*t*	*p*
ALIC	28.688	2.717	26.969	3.800	2.281	.025
ALBC	20.813	4.425	17.359	3.810	3.964	.000
CKT	51.625	6.880	46.422	6.939	3.473	.001
CWNC	32.313	4.490	27.063	4.777	5.177	.000
CP	59.281	8.884	49.359	10.534	4.574	.000

Correlations between the feature scores were analysed separately within each teacher group. Table 4.4 and Table 4.5 show that the five features of teachers' professional development experience were all highly correlated with each other. Comparing the two tables, the correlations between the features in the BE MATHS group were all significantly higher than those in the control group. Because of the strong correlations between the feature scores, it is impossible to determine exactly which feature has the greatest influence on teachers' efficacy scores or students' academic engaged time; moreover, it would not be ethically correct to control the features known to be good for teachers.

Table 4.4
Pearson Correlation Coefficients for Control Group PD Experience Evaluation Scores

Variable	1	2	3	4	5
1. ALIC	1	.531**	.605**	.672**	.436**
2. ALBC	.531**	1	.707**	.813**	.679**
3. CKT	.605**	.707**	1	.764**	.660**
4. CWNC	.672**	.813**	.764**	1	.669**
5. CP	.436**	.679**	.660**	.669**	1

**. Correlation is significant at the .01 level (2-tailed).

Table 4.5
Pearson Correlation Coefficients for BE MATHS Group PD Experience Evaluation Scores

Variable	1	2	3	4	5
1. ALIC	1	.647**	.578**	.577**	.496**
2. ALBC	.647**	1	.722**	.654**	.678**
3. CKT	.578**	.722**	1	.862**	.846**
4. CWNC	.577**	.654**	.862**	1	.900**
5. CP	.496**	.678**	.846**	.900**	1

**. Correlation is significant at the .01 level (2-tailed).

In conclusion, the professional development activities in the BE MATHS programme differ from the activities that in-service teachers usually experience in terms of all five core features. The data showed that all the features of the professional development experience of special school teachers were highly correlated. The correlations between features in the programme were higher than those for the features of a typical professional development experience.

4.3 Mathematics Teaching Efficacy Beliefs

To understand the effect of the BE MATHS programme on teachers' mathematics teaching efficacy beliefs and its contributing factors, quantitative data on teachers' beliefs regarding the mathematics teaching efficacy were collected through their responses to the MTEBI survey. The variables of interest for data analysis are listed as follows:

- *Posttest PMTE*: a continuous variable that is teacher's PMTE score in the posttest;
- *Pretest PMTE*: a continuous variable that is teachers' PMTE score in the pretest;
- *Posttest MTOE*: a continuous variable that is teachers' MTOE score in the posttest;
- *Pretest MTOE*: a continuous variable that is teachers' MTOE score in the pretest;
- *Participation*: a categorical variable that takes a value of 1 for teachers who participated in the BE MATHS programme and a value of 0 for teachers who did not participate;
- *Teaching students with mild ID: a* categorical variable that takes a value of 1 for teachers with mild ID students and a value of 0 for teachers without such students;
- *Teaching students with moderate ID:* a categorical variable that takes a value of 1 for teachers with moderate ID students and a value of 0 for teachers without such students;

- *Teaching students with severe ID:* a categorical variable that takes a value of 1 for teachers with severe ID students and a value of 0 for teachers without such students;
- *Mathematics education background:* categorical variable that takes a value of 1 for teachers who have received specialised mathematics training and a value of 0 for teachers that have not received such training; and
- *Years of teaching:* a categorical variable that takes a value of 1 for teachers with 10 years or more of special school teaching experience and a value of 0 for teachers with less than 10 years of experience.

An independent-samples *t*-test was conducted to make an initial comparison between the PMTE and MTOE scores of the two teacher groups (see Table 4.6). The results ($t(94)=3.356$, $p=.001$) indicated a significant difference in the posttest MTOE scores for the BE MATHS group ($M=31.53$, $SD=1.685$) and the control group ($M=30.17$, $SD=1.956$). This suggests that the participation in the BE MATHS programme has the effect of helping teachers become more confident that their mathematics teaching will lead to successful teaching outcomes among their students. In addition, a paired-samples *t*-test was conducted to compare the pretest and posttest subscale scores for the two groups. As Table 4.7 shows, the PMTE scores for BE MATHS group teachers were better on average in the posttest ($M=51.88$, $SD=3.722$) than in the pretest ($M=51.34$, $SD=3.117$); however, the mean improvement of .531 points was not statistically significant ($t(31)=1.195$, p=.241). Conversely, the MTOE scores for BE MATHS group teachers were significantly better on average in the posttest ($M=31.53$, $SD=1.685$) than in the pretest ($M=30.75$, $SD=2.476$), with the improvement of .781 points being statistically significant ($t(31)=2.352$, $p=.025$). These results suggest that the BE

MATHS programme has a significant effect on teachers' MTOE.

Table 4.6
Descriptive Statistics and Independent-Sample t-Test Results for Pre- and Posttest PMTE and MTOE scores

	BE MATHS group (N=32)		Control group (N=64)		Test results	
MTEBI subscale	*M*	*SD*	*M*	*SD*	*t*	*p*
Pretest PMTE	51.34	3.117	50.80	3.178	.80	.426
Posttest PMTE	51.88	3.722	51.11	2.987	1.089	.279
Pretest MTOE	30.75	2.476	30.28	2.119	.965	.337
Posttest MTOE	31.53	1.685	30.17	1.956	3.356	.001

Table 4.7
Paired-sample t-Test Results Comparing Pre- and Posttest PMTE and MTOE Scores

Score pairs	Teacher group	*M*	*SD*	*SE*	*t*	*df*	*p*
Posttest PMTE – Pretest PMTE	BE MATHS group	.531	2.514	.444	1.195	31	.241
	Control group	.313	3.299	.412	.758	63	.451
Posttest MTOE – Pretest MTOE	BE MATHS group	.781	1.879	.332	2.352	31	.025
	Control group	-.109	2.086	.261	-.419	63	.676

Teachers' performance on the efficacy test was a function of many factors. As Freeman (2017) suggested, multiple linear regression offers a practical framework for distinguishing the impact of an intervention on the effects of sample characteristics on test score gains. In this study, the teachers' pretest score is one obvious control variable, as it presents each of their prior knowledge and efficacy levels. Other factors—the participant's mathematics training background, their previous experience in teaching, and their students' degrees of disability—also need to be controlled because they may affect a teacher's belief in their teaching efficacy, regardless of the study group to which the teacher belongs. The following two hypotheses were tested to answer RQ2:

- Null Hypothesis 1 (NH1): The BE MATHS programme does not affect special school teachers' PMTE scores after controlling for teachers' background factors.

- Null Hypothesis 2 (NH2): The BE MATHS programme does not affect special school teachers' MTOE scores after controlling for teachers' background factors.

To determine the outcomes for NH1, a multiple linear regression was conducted where the dependent variable, teachers' posttest PMTE scores, was analysed with respect to the independent variable of interest (participation in the BE MATHS programme) and additional variables for controlling participants' demographic differences (Hayes, 2018). The analysis found an interaction between teachers' pretest scores and programme participation (see Table 4.8 and Table 4.9). To probe the interaction, all the continuous variables were mean-centred to facilitate the interpretation of the regression parameters (Hayes, 2018). The overall model ($F(8,87)=6.693$, $p<.001$, $R^2=.381$) was significant.

Table 4.8

Pearson Correlation Coefficients for PMTE Scores, Teacher Type, and Regression Analysis Control Variables

Variable	1	2	3	4	5	6	7	8
1. Posttest PMTE score	1	.547**	.188	–.034	–.239*	.250*	.065	.112
2. Pretest PMTE score	.547**	1	.230*	–.096	–.294	.184	.086	.082
3. Teaching mild ID students	.188	.230*	1	.063	–.389**	.176	–.034	.016
4. Teaching moderate ID students	–.034	–.096	.063	1	–.307**	–.094	.137	–.140
5. Teaching severe ID students	–.239*	–.294**	–.389**	–.307**	1	–.146	.002	.053
6. Mathematics education background	.250*	.184	.176	–.094	–.146	1	–.028	–.016
7. Years of teaching	.065	.086	–.034	.137	.002	–.028	1	–.030
8. Participation	.112	.082	.016	–.140	.053	–.016	–.030	1

**. Correlation is significant at the .01 level (2-tailed).

*. Correlation is significant at the .05 level (2-tailed).

Table 4.9
Regression Analysis Results for Posttest PMTE Scores

Dependent variable—posttest PMTE score				
	b	*SE*	*t*	*p*
Control variables				
Teaching students with mild ID	.353	.648	.544	.588
Teaching students with moderate ID	.119	.634	.188	.851
Teaching students with severe ID	–.221	.797	–.277	.783
Mathematics education background	1.065	.616	1.728	.088
Years of teaching	.015	.567	.027	.979
Independent variable				
Participation	.433	.589	.736	.464
Moderator				
PMTE pretest score (centred)	.353	.112	3.158	.002
Interaction term				
PMTE pretest score (centred) × participation	.489	.191	2.561	.012

As the interaction (b=.489, t(87)=2.561, p=.012) was significant, it was probed by conducting a test of simple slopes using the Johnson-Neyman technique. Figure 4.2 and Table 4.10 show that the relationship between participation in the BE MATHS programme and teachers' overall posttest MTEBI scores was significant when teachers' pretest MTEBI score was one standard deviation above the mean (p=.017) but was not significant with lower pretest scores. These results suggest that participating in the BE MATHS programme helps teachers whose pretest MTEBI scores are above the mean achieve higher mathematics teaching efficacy by the end of the school year.

Figure 4.2
Simple Slopes of Participation Predicting Posttest MTEBI Scores for Various Pretest MTEBI Scores

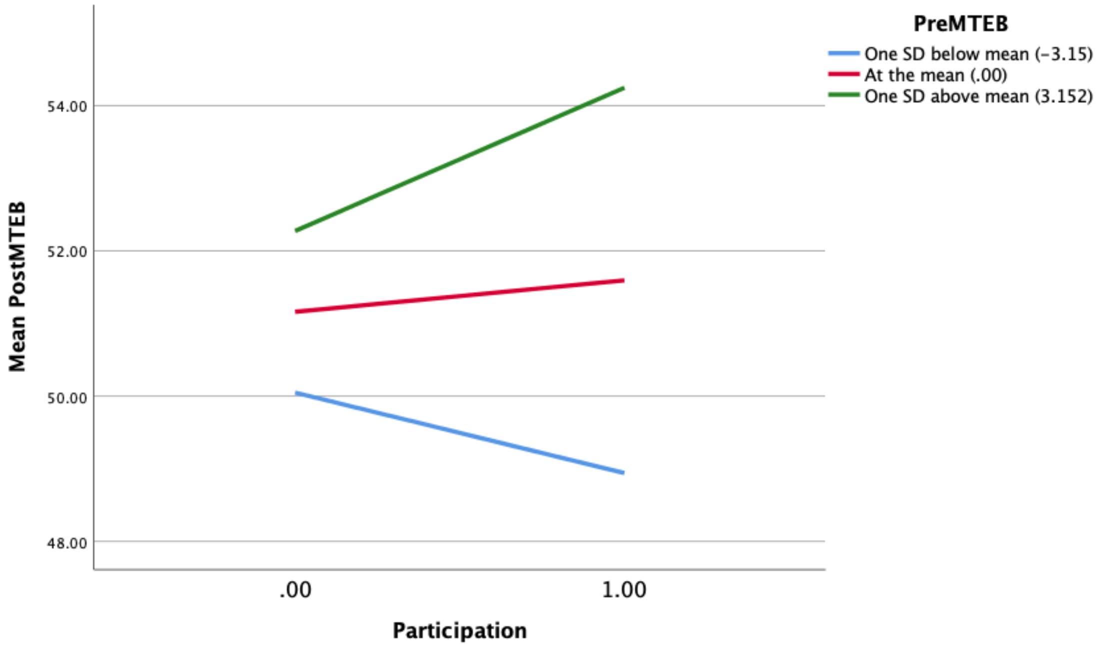


Table 4.10
Conditional Effects of BE MATHS Programme Participation on Pretest MTEBI Scores

Pretest MTEBI score	*β*	*SE*	*t*	*p*
One *SD* above mean (3.152)	1.974	.809	2.439	.017
At the mean (.000)	.433	.589	.736	.464
One *SD* below mean (–3.152)	–1.108	.873	–1.269	.208

To test NH2, using the same analytical approach as that used for the PMTE scores, teachers enrolled in the BE MATHS programme were found to have significantly higher posttest MTOE scores compared with control group teachers (Table 4.11 and Table 4.12). The regression coefficient (b=1.214) for the posttest MTOE scores indicated that BE MATHS group teachers scored 1.214 points higher than control group teachers when all other demographic factors were controlled (t=3.471, p=.001). The presence of an interaction effect was also investigated, but none was found.

Table 4.11

Pearson Correlation Coefficients for MTOE Scores, Teacher Type, and Regression Analysis Control Variables

Variable	1	2	3	4	5	6	7	8
1. Posttest MTOE score	1	.531**	.084	–.038	–.002	.140	–.093	.327**
2. Pretest MTOE score	.531**	1	.096	–.202*	–.070	.058	–.113	.099
3. Teaching mild ID students	.084	.096	1	.063	–.389**	.176	–.034	.016
4. Teaching moderate ID students	–.038	–.202	.063	1	–.307**	–.094	.137	–.140
5. Teaching severe ID students	–.002	–.070	–.389**	–.307**	1	–.146	.002	.053
6. Mathematics education background	.140	.058	.176	–.094	–.146	1	–.028	–.016
7. Years of teaching	–.093	–.113	–.034	.137	.002	–.028	1	–.03
8. Participation	.327**	.099	.016	–.140	.053	–.016	–.03	1

**. Correlation is significant at the .01 level (2-tailed).

*. Correlation is significant at the .05 level (2-tailed).

Table 4.12
Regression Analysis Results for Posttest MTOE Scores

Dependent variable—posttest MTOE score				
	b	*SE*	*t*	*p*
Control variables				
Teaching students with mild ID	.461	.076	6.085	0
Teaching students with moderate ID	.137	.382	.358	.721
Teaching students with severe ID	.645	.379	1.702	.092
Mathematics education background	.475	.455	1.044	.3
Years of teaching	.585	.365	1.603	.112
pretest MTOE score	.461	.076	6.085	0
Independent variable				
Participation	1.214	.35	3.471	.001

Overall, participation in the BE MATHS programme increased teachers' mathematics teaching efficacy given the prior condition that their pretest MTEBI scores were above the participants' average. Participation in the programme also significantly improved teachers' MTOE scores.

4.4 Student Academic Engaged Time

After collecting the video recordings, a team of six reviewers started the arduous task of analysing them for learning engagement behaviours. All reviewers had prior teaching or research experience. Over the first two months, all the reviewers studied the manual for the ICER scale as they watched and discussed the recordings together, aiming to reach a consensus on describing students' engagement. In order to avoid biasedness, the reviewers were not informed about which group (control group and BE MATHS group) that a video belongs to. Four types of engagement were intended to be coded during video analysis. The definitions for these types given by the ICER manual are as follows:

- *Active engagement* (AE): The child actively participated in the activity by interacting with the learning environment appropriately through manipulating materials or vocalising. The child did not demonstrate repetitive behaviours.
- *Passive engagement* (PE): The child interacted with the environment without manipulation or vocalisation.
- *Active nonengagement* (AN): The child interacted with the environment inappropriately through manipulation, movement and/or vocalisation.
- *Passive nonengagement* (PN): The child did not interact with the environment and did not do what was expected of them during the activity.

While the team of reviewers reached an agreement on students' active behaviours, the reviewers had differences of opinion regarding **passive** behaviours. Although the definitions for passive engagement and passive nonengagement sounded sensible, the reviewers quickly found that these definitions could not be applied during the coding process. Because passive behaviours were only defined based on when a student did not exhibit manipulation or vocalisation, without other observable evidence, a certain amount of reviewer subjectivity was involved when coding for passive behaviour. For instance, there was a case where a student looked toward where his teacher was teaching. According to the ICER manual's definitions, such behaviour should be an example of passive engagement; however, some observers disagreed with the judgement, as they found that after several minutes, the student was still looking in that direction even when his teacher had left. Without movements or vocalisation, it was impossible to know for certain whether a student was actually paying attention to the teacher's instruction or just daydreaming or pretending to be engaged in learning. Consequently, in certain circumstances that were observed in the actual study, the definitions of passive engagement and nonengagement simply did not apply. This finding was important because it meant that it was fruitless to try interpreting the coding results for passive engagement and nonengagement if the results were inconsistent with the definitions for such behaviour. After careful deliberation, all the team members abandoned coding for passive engagement and passive nonengagement and instead only coded for active engagement and active nonengagement.

The variables of interest for data analysis are listed as follows:

- *AE1*: a continuous variable that measures the percentage of intervals where students' active engagement behaviours were observed during the first lesson observation;
- *AE2*: a continuous variable that measures the percentage of intervals where students' active engagement behaviours were observed during the second lesson observation;
- *AE3*: a continuous variable that measures the percentage of intervals where students' active engagement behaviours were observed during the third lesson observation;
- *AN1*: a continuous variable that measures the percentage of intervals where students' active nonengagement behaviours were observed during the first lesson observation;
- *AN2*: a continuous variable that measures the percentage of intervals where students' active nonengagement behaviours were observed during the second lesson observation;

- *AN3*: a continuous variable that measures the percentage of intervals where students' active nonengagement behaviours were observed during the third lesson observation; and
- *Participation*: a categorical variable that takes a value of 1 for teachers who participated in the BE MATHS programme and a value of 0 for teachers who did not participate.

An independent-sample *t*-test was conducted to compare the means of the three AE variables (AE1, AE2, AE3) for the students of teachers in both study groups. Table 4.13 shows that the AE values for both groups of students differed significantly across all three observation sessions.

Table 4.13
Descriptive Statistics and Independent-Sample t-Test Results for Active Engagement

	BE MATHS group			Control group			Test results	
Variable	*N*	*M*	*SD*	*N*	*M*	*SD*	*t*	*p*
AE1	54	.397	.214	27	.230	.155	3.610	.001
AE2	51	.396	.163	27	.194	.134	5.538	.000
AE3	49	.468	.167	26	.198	.135	7.107	.000

Regression techniques were applied to control for the differences in AE1 between the two groups of students at the first (baseline) lesson observation. A multiple linear regression was conducted where the dependent variable, i.e., the proportion of time that students were actively engaged in mathematics class, was analysed with respect to the independent variable of interest (teacher's participation in the BE MATHS programme) and two additional variables for controlling differences in students' behaviours (the proportion of time that students were actively engaged during the first lesson observation and student ID level). Table 4.14 and Table 4.15 list the correlation coefficients between the variables and the regression coefficients for the variables, respectively.

Table 4.14
Pearson Correlation Coefficients for AE2 Scores, Teacher Type, and Control Variables

Variable	1	2	3	4	5
1. AE2	1.000	.742**	.054	–.144	.536**
2. AE1	.742	1.000	.092	–.145	.376**
3. Mild ID	.054	.092	1.000	–.432**	–.163
4. Severe ID	–.144	–.145	–.432**	1.000	.070
5. Participation	.536**	.376**	–.163	.070	1.000

**. Correlation is significant at the .01 level (2-tailed).

Table 4.15
Regression Analysis Results for AE2 Scores

Dependent variable—AE at 2nd observation (AE2)				
	b	*SE*	*t*	*p*
Control variables				
Student with mild ID	.011	.029	.375	.709
Student with severe ID	–.053	.039	–1.367	.176
Independent variable				
Participation	.060	.035	1.742	.086
Moderator				
AE1 (centred)	.892	.158	5.652	.000
Interaction term				
AE1 (centred) × participation	–.449	.176	–2.547	.013

As Table 4.15 shows, students' AE1 scores significantly moderated the relationship between their teachers' participation in the BE MATHS programme and their AE2 scores. Figure 4.3 illustrated this interaction using simple slopes. The interaction was further probed by testing the conditional effects of BE MATHS programme participation at three levels of AE1 scores, namely, at the mean and one standard deviation away on both sides. As Table 4.16 shows, teachers' programme participation significantly increased their students' AE2 scores when the students' AE1 scores were one standard deviation below the mean ($p<.001$) but not when the AE1 scores were either at the mean or one standard deviation above it (p=.086 and p=.573, respectively). The Johnson-Neyman technique also showed that programme participation had a significant positive impact on students' AE2 scores when students' AE1 scores were below .327 points (.0147 standard deviations below the mean, t=1.994, p=.05, b=.067), which was not significant with higher AE1 scores.

Figure 4.3
Simple Slopes of Participation Predicting AE2 Scores for Various AE1 Scores

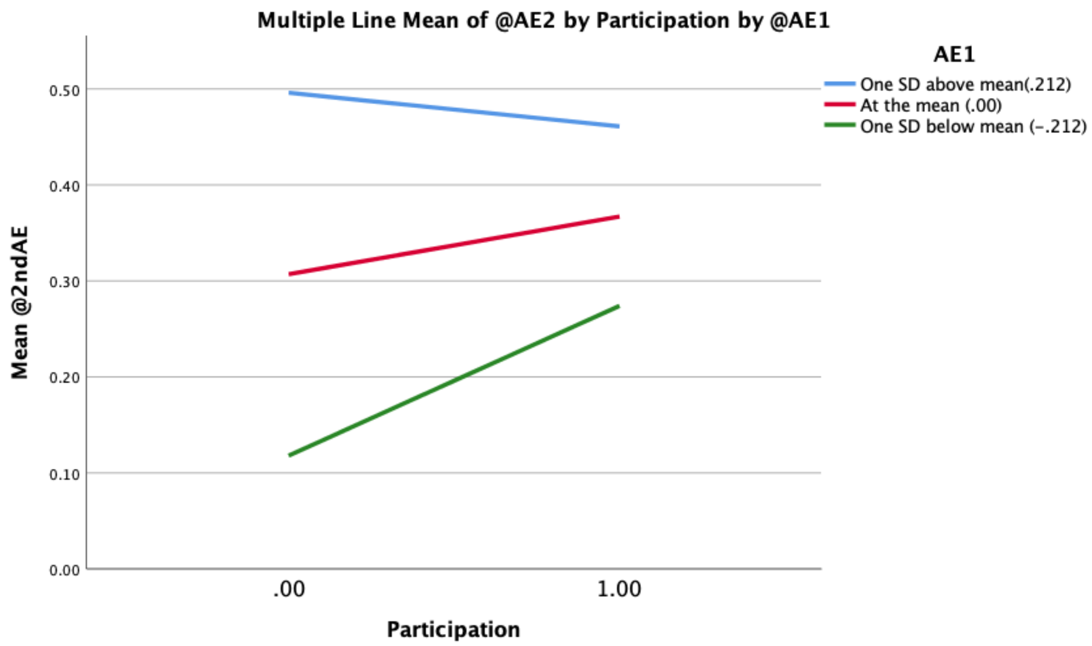


Table 4.16
Conditional Effects of BE MATHS Programme Participation on AE2 Scores

AE1 score	*β*	*SE*	*t*	*p*
One *SD* above mean (.212)	–.035	.061	–.566	.573
At the mean (.000)	.060	.035	1.742	.086
One *SD* below mean (–.212)	.156	.038	4.082	.000

Applying the same approach to analysing the effect of teachers' BE MATHS programme participation on students' AE3 scores (see Table 4.17 for descriptive statistics), the regression coefficient table (Table 4.18) shows that students' AE1 scores also moderated the effect of programme participation on AE3 scores, as illustrated in Figure 4.4 using simple slopes. Table 4.19 shows that teachers' programme participation significantly increased their students' AE3 scores when the students' AE1 scores were at the mean or one standard deviation below it (p<.001) but not when the AE1 scores were one standard deviation above the mean (p=.111). Further analysis using the Johnson-Neyman technique showed that when students' AE1 scores were below .517 points (.176 above the mean), programme participation had a significant positive impact on students' AE3 scores (t=1.996, p=.05, b=.108).

Table 4.17
Pearson Correlation Coefficients for AE3 Scores, Teacher Type, and Control Variables

Variable	1	2	3	4	5
1. AE3	1.000	.715**	.079	–.054	.640**
2. AE1	.715**	1.000	.092	–.145	.376**
3. Mild ID	.079	.092	1.000	–.432**	–.163
4. Severe ID	–.054	–.145	.432**	1.000	.070
5. Participation	.640**	.376**	–.163	.070	1.000

**. Correlation is significant at the .01 level (2-tailed).

Table 4.18
Regression Analysis Results for AE3 Scores

Dependent variable—AE at 3rd observation (AE3)				
	b	*SE*	*t*	*p*
Control variables				
Student with mild ID	.057	.031	1.838	.071
Student with severe ID	.019	.040	.487	.628
Independent variable				
Participation	.170	.034	4.930	.000
Moderator				
AE1 (centred)	.795	.155	5.142	.000
Interaction term				
AE1 (centred) × participation	–.351	.174	–2.022	.047

Figure 4.4
Simple Slopes of Participation Predicting AE3 Scores for Various AE1 Scores

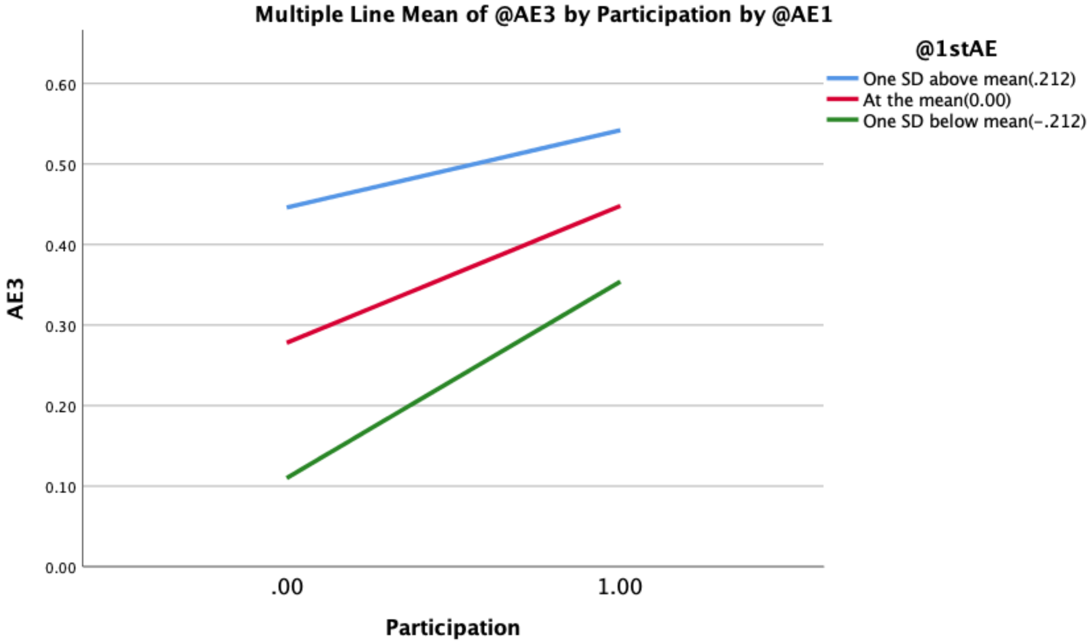


Table 4.19
Conditional Effects of BE MATHS Programme Participation on AE3 Scores

AE1 score	*β*	*SE*	*t*	*p*
One *SD* above mean (.212)	.095	.059	1.615	.111
At the mean (.000)	.170	.034	4.930	.000
One *SD* below mean (–.212)	.244	.040	6.137	.000

A comparison of the regression analysis results for AE2 and AE3 scores found that the BE MATHS programme can help teachers make their students who rarely engage in mathematics classes become more actively engaged in mathematics learning. However, for students who were already engaged in mathematics classes before their teachers participated in the programme, the programme did not have a significant impact on their learning engagement.

Turning to the analysis of the effect of teachers' BE MATHS programme participation on students' AN, an independent-sample *t*-test comparing the means of the AN variables (AN1, AN2 and AN3) found no significant differences in the AN values for both groups of students across the three observation sessions (Table 4.20). Regression techniques were then applied to control for the demographic differences between both groups when analysing the students' AN2 scores (see Tables 4.21 and 4.22).

Table 4.20
Descriptive Statistics and Independent-Sample t-Test Results for Active Nonengagement

	BE MATHS group			Control group			Test results	
Variable	*N*	*M*	*SD*	*N*	*M*	*SD*	*t*	*p*
AN1	54	.220	.213	27	.249	.152	-.639	.525
AN2	51	.194	.216	27	.222	.169	-.801	.425
AN3	49	.145	.157	26	.190	.125	-1.256	.213

Table 4.21
Pearson Correlation Coefficients for AN2 Scores, Teacher Type, and Control Variables

Variable	1	2	3	4	5
1. AN2	1.000	.738	–.120	.378**	–.092
2. AN1	.738	1.000	–.019	.178	–.072
3. Mild ID	–.120	–.019	1.000	–.432**	–.163
4. Severe ID	.378**	.178	–.432	1.000	.070
5. Participation	–.092	–.072	–.163	.070	1.000

**. Correlation is significant at the .01 level (2-tailed).

Table 4.22
Regression Analysis Results for AN2 Scores

Dependent variable—AN at 2nd observation (AN2)				
	b	*SE*	*t*	*p*
Control variables				
Student with mild ID	–.008	.034	–.243	.809
Student with severe ID	.117	.047	2.483	.015
AN1	.706	.081	8.700	.000
Independent variable				
Participation	–.024	.033	–.746	.458

Table 4.20 suggests that programme participation had no significant effect on students' AN2 scores. However, for the AN3 scores representing the third observation session,

the regression coefficient table (Table 4.24; see also Table 4.23) illustrates that programme participation resulted in a significant decrease in students' AN behaviours (*b*=-.054, *t*=-2.502, *p*=.015). When controlling for the differences in AN1 and students' ID levels, the active non-engaged time of a student whose teacher participated in the BE MATHS programme decreased by 5.4% by the third observation session. No interaction effect was found in the regression analyses for either AN2 or AN3 scores.

Table 4.23
Pearson Correlation Coefficients for AN3 Scores, Teacher Type, and Control Variables

Variable	1	2	3	4	5
1. AN3	1.000	790**	–.227	267*	–.146
2. AN1	790**	1.000	–.019	–.178	–.072
3. Mild ID	–.227	–.019	1.000	–.432**	–.163
4. Severe ID	–.267*	–.178	–.432**	1.000	–.070
5. Participation	–.146	–.072	–.163	–.070	1.000

**. Correlation is significant at the .01 level (2-tailed).
*. Correlation is significant at the .05 level (2-tailed).

Table 4.24
Regression Analysis Results for AN3 Scores

Dependent variable—AN at 3rd observation (AN3)				
	b	*SE*	*t*	*p*
Control variables				
Student with mild ID	–.057	.023	–2.510	.015
Student with severe ID	.016	.029	.558	.579
AN1	.576	.052	11.090	.000
Independent variable				
Participation	–.054	.021	–2.502	.015

In conclusion, participation in the BE MATHS programme can help teachers increase their students' academic engaged time in mathematics classes if students were previously rarely engaged in class. Programme participation can also lead to decreases in students' active non-engaged time.

Chapter 5 Qualitative Findings

The previous chapter presented the quantitative findings based on the data collected through online surveys and lesson observations for students' academic engaged time and illustrated the changes in teachers' and students' behaviours following teachers' participation in the BE MATHS programme. This chapter provides the results from an in-depth analysis of qualitative data collected through interviews and teacher observations and explores the contributing factors for the changes. Specifically, this chapter answers RQ4: "How does the BE MATHS programme influence teachers' mathematics teaching?"

The chapter begins with a review of the general findings that emerged from the qualitative data collection exercises. It continues by presenting an analysis of three cases involving teachers teaching students with mild, moderate, and severe ID, aiming to deepen the understanding gained from the general findings. The study ultimately interviewed 12 teachers from 12 special schools in Hong Kong, including four teachers from schools for mild ID students, five teachers from schools for moderate ID students, and three teachers from schools for severe ID students (Table 5.1). Observations were also made at 40 meetings, including 20 learning circle collaborative planning meetings lasting approximately 1.5–2 hours each and 20 post-lesson observation meetings lasting approximately 1 hour each.

Table 5.1
Descriptive Statistics for Interviewed Teachers (N=12)

Descriptor	Statistics
Age (yrs.)	*M*=37.917, *SD*=9.366 (range: 26—55)
Teaching experience (yrs.)	*M*=12.167, *SD*=8.569 (range: 2—26)
Students ID level	Mild: *n*=4 (33.3%); moderate: *N*=5 (41.7%); severe: *n*=3 (25%)

5.1 Learning Teaching Through Studying *Hypothetical Teaching and Learning Trajectories*

When asked how the BE MATHS programme improved their teaching during interviews, all the teachers expressed that they learned a lot from studying the "backbone design [*gwat gaa fong on*]", which is a term that was frequently mentioned by teachers during interviews and frequently heard during the meeting observations. One of the subject advisors defined "backbone design" as "a framework for the organisation of the teaching unit and its major features" (Fung, 2016, p. 20). Nevertheless, what the teachers meant by "backbone design" could be more accurately described as a *hypothetical teaching and learning trajectory*. A "backbone design" is not only a framework but also a detailed progressive learning trajectory with many clearly described intermediate learning stages and concrete descriptions of teaching activities for each stage. It also contains practical advice such as the selection of manipulatives and instructional words, phrases, and sentences, in addition to alternative learning targets for students with higher or lower learning abilities, or with other SENs such as language impairment.

Participating teachers and subject advisors developed these hypothetical teaching and learning trajectories together in learning circle meetings. The developed trajectories were documented using texts and figures to serve as curriculum resources for the teachers' easy reference. These documents will also be passed onto the next iteration of the learning circle interested in the same topic. Although the fundamental structure of the trajectory will not change, the new cohort of teachers can have their input in certain elements of the design, thus contributing refinements and add-ons where appropriate.

Identifying and Bridging Knowledge Gaps

When talking about how the hypothetical teaching and learning trajectories facilitate teaching, one theme that teachers mentioned the most was identifying and bridging *knowledge gaps*. Teachers mainly referred to this concept using expressions such as "knowing more about learning trajectory", "becoming clearer about the knowledge sequence/concept", "understanding what I have missed in my teaching", "I have a better conceptual understanding of mathematics", and "finding out the intermediate steps".

A knowledge gap refers to something that is missing from a teacher's expertise or skills that prevents them from providing a complete or satisfactory learning experience to their students, as illustrated by the following comment:

> "Let's say an (ID) student learning a concept takes four steps. Before joining this programme, I only knew step 1 and step 4, and I omitted steps 2 and 3 in my teaching. Now I know there are steps between step 1 and step 4, and only when my students take all these steps will they understand the concept."

The above comment touches on the idea that some parts of learning trajectories essential to ID students are not well understood by their teachers. There is thus a disparity between the students' teaching needs and the teachers' knowledge. One teacher believed that the occurrence of this difference was normal for a teacher of ID students:

> "The knowledge is the blind points that I have not realised before. Because, as I am a person with a normal intelligence, I had no experience learning concepts in such small steps as those in which my students learn. It is natural that I have missed all these small steps when I teach my students because my learning trajectory didn't contain them!"

From the teacher's perspective, these gaps were identified and bridged when they studied the *hypothetical teaching and learning trajectories* in the programme. As another teacher put it:

> "After studying the backbone design, I think I understand why my students failed to learn a certain concept before. It is because I omitted some content in my teaching." "The backbone design lists all the small steps out. When we read it, we understand what we have missed in our teaching. In other words, it improves our profession."

This comment shows that the intermediate learning steps in the *hypothetical teaching and learning trajectories* make up for the teachers' lack of knowledge, thereby helping them understand how an ID student can possibly master a new concept.

<u>Systematic Instructions</u>

Another theme of the interviews was that the *hypothetical teaching and learning trajectories* help teachers "teach more *systematically*". Teachers' use of the term "systematic" expressed two related meanings: "detailed" and "organised".

As students with different learning abilities learn at different paces, the more severe a student's ID is, the more *detailed* the picture of how the student learns needed to be. For teachers of ID students, the learning contents suggested by curriculum materials or teaching resources for mainstream students are too broad and thus uninformative for teaching. A teacher of moderate ID students used "counting from 1 to 10" as an example to illustrate this phenomenon:

> "These topics [such as counting from 1 to 10] are so easy that people could not even figure out what to teach! However, students with moderate ID spend years on these topics. What to teach and how to teach these topics are difficult questions for us."

Another teacher from a school for severe ID students described a similar situation as follows:

> "Unlike teachers in mainstream schools who have a curriculum and mathematics textbooks to follow, teachers in my school have no idea what to teach! I mean, we know that students need to learn mathematics, but what content of mathematics is appropriate for students with severe ID? How can we design a learning experience that is both mathematically rich and aligns with our students' ability?"

The issue the two teachers referred to—not knowing the appropriate learning content that aligns with the abilities of their target group of students—was also mentioned by other teachers, typically those from schools for students with moderate or severe ID. When certain content—which the teachers believe is at the very beginning of mathematics learning and too simple to be taught—is still beyond the reach of students, it is all too natural that the teachers feel they are at their wits' end. They fail to identify appropriate action to be taken on the teachers' side. Without a clear objective to achieve, teaching is merely a collection of rambling speech and meaningless actions.

All the teachers said the *hypothetical teaching and learning trajectories* were designed in accordance with the specific learning abilities of target students. The learning steps were small enough to enable teachers to understand what they could teach. A teacher of severe ID students commented as follows:

> "I have found that teaching mathematics is not just teaching addition, subtraction, multiplication and division after all. Concepts such as 'something' and 'nothing', 'same' and 'different' can also be within the learning content of a mathematics class. They are the concepts from an earlier stage of mathematics learning, which are aligned with my students' abilities. I have not thought about teaching this content before."

The term *organised* refers to another feature of *hypothetical teaching and learning trajectories* that contributed to teachers' expertise. Four teachers mentioned this term when discussing the role of the trajectories in their professional development. They described the trajectories as being just like a manual for teaching. They inform teachers where a student is in their learning and how the student can step into the next stage through suggested learning activities. T3 illustrated this point in her learning experience:

> "My lesson was just a collection of sensory stimulation activities that seemed to be related to mathematics. It was designed without any organisation. It was common that I taught many concepts in a single lesson, such as heavy, light, big, small, long, short, counting, but without any connections between them."

> "With a backbone design, the teaching content in my lesson is more focused. For example, when teaching big and small, I only focus on big and small. I will not mix other concepts in this lesson. The backbone design contains many levels to learn a single concept. It provides me with a plan where I can find which level a student is, and what is the next level he or she should step into. For example, students need to observe identical objects and learn (the phenomenon of) the 'same' before they can notice (the phenomenon of) the 'difference' in volume. It's as if you have a procedure to follow. You are clear about what the first step is when teaching a concept, and what the next step is."

5.2 The Growth of KCT, SCK and KCS

What is the nature of *hypothetical teaching and learning trajectories*? What types of knowledge are included in the design process for the trajectories? To answer these questions, interview transcripts, observation records, and documented *hypothetical teaching and learning trajectories* were analysed and coded based on the knowledge framework developed by Hill, Ball, et al. (2008) (see Chapter 2, Figure 2.1). To further understand the source of the knowledge, whether the specific knowledge came from the subject advisor or from participating teachers was also coded. Table 5.2 shows the distribution of the coding cases.

Table 5.2

Coding of Knowledge for Mathematics Teaching

Code	Illustration	Case example	Cases	Contributor*
Common content knowledge (CCK)	Mathematical knowledge and skills not unique to teaching.	A student mistakenly recognised 4 as 5, and this is recognised as incorrect.	14	A(6) T(8) A&T(0)
Horizon content knowledge (HCK)	An understanding of how mathematics topics are related to a larger mathematical landscape.	If we understand that any fraction is composed by one or more identical unit fraction(s), we can solve problems of fractions by using the same kind of thinking we used in integers.	6	A(4) T(2) A&T(0)
Specialised content knowledge (SCK)	Mathematical knowledge not typically needed for purposes other than teaching.	The way to divide two integers without using multiplication.	94	A(87) T(7) A&T(0)
Knowledge of content and students (KCS)	Knowledge about what content of mathematics students are likely to think about and what content of mathematics students will find confusing.	When being asked to count out a stated number of objects, many students with moderate ID will not stop counting when he or she has reached that number.	66	A(21) T(45) A&T(0)
Knowledge of content and teaching (KCT)	Combines knowing about teaching and knowing about mathematics.	When teaching clock, using a geared hands clock could help students find that one hour is equal to sixty minutes.	142	A(32) T(62) A&T(48)
Knowledge of curriculum (KC)	Broad comprehension of school subjects, awareness of instructional materials, learning objectives and knowledge of what has been previously taught in one's subject and what will be taught in the future.	Providing students with enough experience in grouping and counting items by 10s will help student learn place values more easily in their later school years.	8	A(4) T(4) A&T(0)
General pedagogical knowledge (GPK)	Knowledge of teaching methods, assessment, lesson structure, and knowledge of various cognitive and motivational learning theories, strategies, and knowledge of individual student characteristics.	Using multi-sensory strategy can make students be more focused in learning.	14	A(3) T(11) A&T(0)

*A(): the number of cases in which knowledge is first contributed/identified/used/introduced by advisors; T(): the number of cases in which knowledge is first contributed/identified/used/introduced by teachers; A&T(): the number of cases in which knowledge is developed by teachers and advisors together.

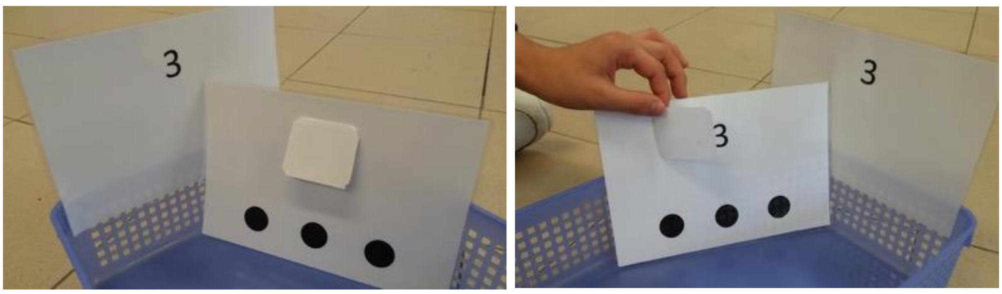
The above table shows that KCT (142 cases), SCK (94 cases), and KCS (66 cases) are the three domains of knowledge mentioned the most frequently in the programme.

KCT is the domain of knowledge that teachers discussed the most during the meeting of learning circles. Five themes emerged in the cases of KCT:

1. **Knowledge about tools and learning activities that make a specific mathematics content comprehensive to ID students.** Many of the discussions in meetings were centred around this theme. It includes, among others, discussions such as those about the selection of teaching aids, the selection of vocabulary used in teaching, and the way to communicate an idea. For example, teachers found that some students with moderate ID can count using a group of counters but cannot link the result with number symbols. The learning circle designed a transition card (see Figure 5.1) that contains both the symbol and the counters. Students can use the transition card to link the card of the counter to the card of the number symbol. This transition card serves as an instruction tool for teaching the symbol "3" and represents the quantity "3". However, when students are familiar with the relationship, the transition card will be removed from the matching activity.

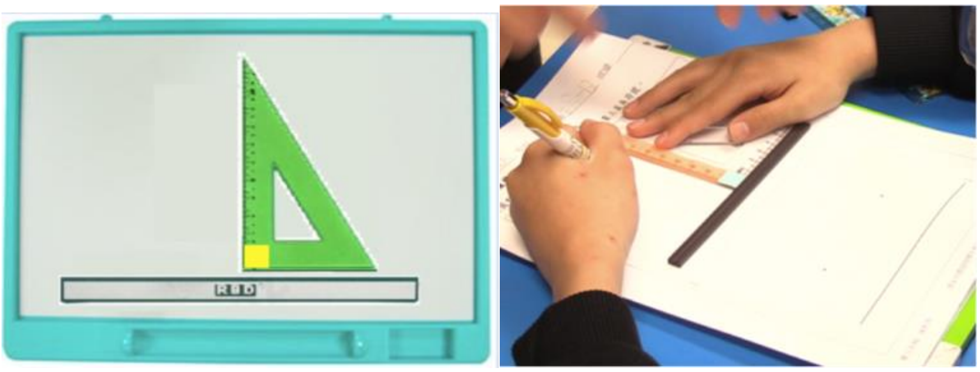
Figure 5.1

Transition Card Linking the Quantity "Three" to the Number Symbol "3"



2. **Designing tools or aids to assist students in mathematics.** This theme always occurs when there are some important learning tasks that students need to complete, but teachers believe their students will struggle with it because of some physical or cognitive problems. For example, teachers predicted that their students would have great difficulty drawing a perpendicular line through a given point because it involves synchronising both hands and eyes, and most of their students are not good at the coordination of small muscles. Figure 5.2 shows the assistive tools designed by the learning circle. With the help of the magnetic whiteboard with a long magnetic stripe, it is easy for students to "press" the straight stripe tightly so that they can focus on setting another ruler perpendicular to the stripe and through the given point.

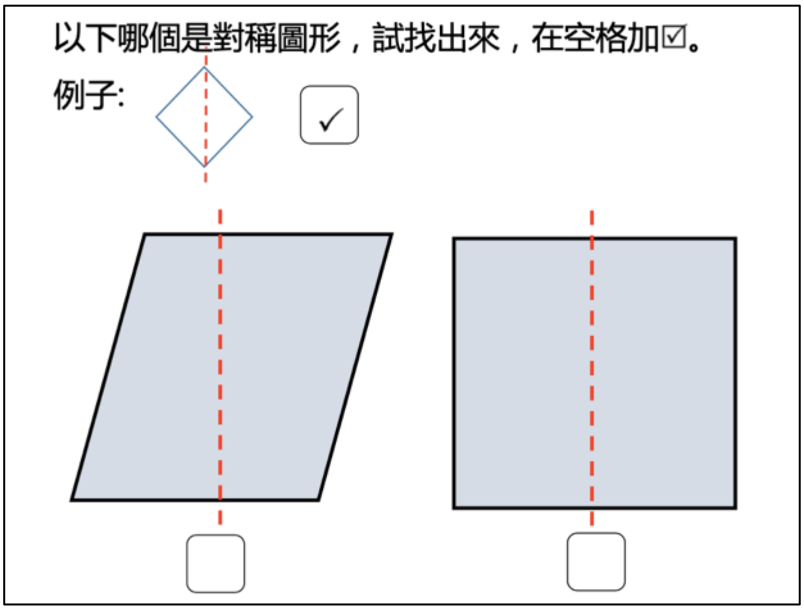
Figure 5.2
Assistive Tools for Drawing Perpendicular Lines



3. **Varying the level of guidance in helping students complete a mathematical task**. ID students were described as being always lost when completing a task with multiple steps. To help students gain confidence in mathematics tasks, teachers provide guidance to them. However, to what extent should guidance be provided? What kind of guidance should be provided? And what should be

avoided? These are the questions discussed by teachers and advisors. For example, in a discussion about teaching mild ID students to figure out shapes with axial symmetry, a teacher suggested providing lines as hints (see Figure 5.3) to lower the difficulty of the task. During the discussion, the subject advisor pointed out that it is problematic to provide hints of lines in this task. When proving that a figure is symmetrical, one needs to find an axis that divides the figure into two mirror figures. However, when proving a figure is not symmetrical, one needs to prove no axis of symmetry exists on the figure. This means that they must test every possible line on the figure and find that all these lines are not the axis of symmetry. The worksheet designed by the teacher helps students to make a claim that relies only on casual testing, which is not objective or scientific.

Figure 5.3
Inappropriate Hints for Teaching Axial Symmetry of Shapes



(In English: Determine whether each shape below is axially symmetric. If so, put a '✓' in the suitable box. Example.)

4. **Selection of vocabularies or sentences that students need to learn.** Developing the language of mathematics is also frequently discussed by learning circles. In most cases, language concerns are first mentioned by the subject advisor and then agreed with by teachers. Such concerns include two aspects. One is about how teachers could use mathematics language in a precise and systematic way. The other is about how to build students' mathematical vocabulary effectively. In teaching resources developed by teachers from special schools for severe ID students, they carefully write out the vocabulary issue on that topic. They pointed out that teachers always unintentionally use different vocabularies to describe one thing. For example, when describing an object's size, teachers use words such as "big," "large," and "huge." Students with normal intelligence quickly understand that these words have the same meaning, but ID students, especially those at moderate and severe levels, feel very confused. Another example is the use of "how much money [*gei do chin*]?" in mathematics lessons. In Chinese, when you say "*gei do chin*", it could mean that you are asking for either "the value of a coin", "the total value of coins" or "the number of coins". Therefore, using "*gei do chin*" in a mathematics lesson would be unclear, especially for ID students. During the discussion of teaching money, teachers learned that they needed to express what they mean more clearly and teach students three different words to represent the three different meanings.

5. **The process for abstracting mathematics content from manipulatives**. Teachers and subject advisors also spent much time discussing the design of mathematics activities so that students could gradually develop a mathematical

understanding of them. Table 5.3 shows a design for teaching students with mild ID to find the total value of two coins. The learning process is arranged from concrete to abstract and is closely linked with students' learning experience of addition.

Table 5.3
Learning Levels for the Total Value of Two Coins

Level 1	 (In English: a combination of [picture: a 5-dollar coin] and [picture: a 2-dollar coin], making a total of __7__ dollars)
Level 2	 (In English: a combination of [picture: a 5-dollar coin] and [picture: a 2-dollar coin], making a total of __7__ dollars)
Level 3	 (In English: a combination of [picture: a 5-dollar coin] and [picture: a 2-dollar coin], making a total of __7__ dollars)
Level 4	 (In English: a combination of [picture: a 5-dollar coin] and [picture: a 2-dollar coin], making a total of __7__ dollars)

As for the SCK, the following four themes were identified:

1. **Knowledge about the components of a mathematics concept or procedure.** During the meetings, the components of a mathematics concept/procedure were always analysed. For example, a teacher reported that her students cannot count

out a specific number of items, and the advisor illustrated the components of the task to teachers as follows: "It means that a child needs to know the conventional sequence of the number names, be able to match these names to the words one by one, be able to monitor whether the number name tagged to an object has reached the requested number when he is counting and know that he or she needs to stop counting when the number has been reached." This kind of knowledge enabled teachers to have a clear picture of what students need to learn on this topic. Compared to the learning activities they designed for their students, teachers quickly determined that they had not built up their students' awareness of "stop counting when the requested number has been reached."

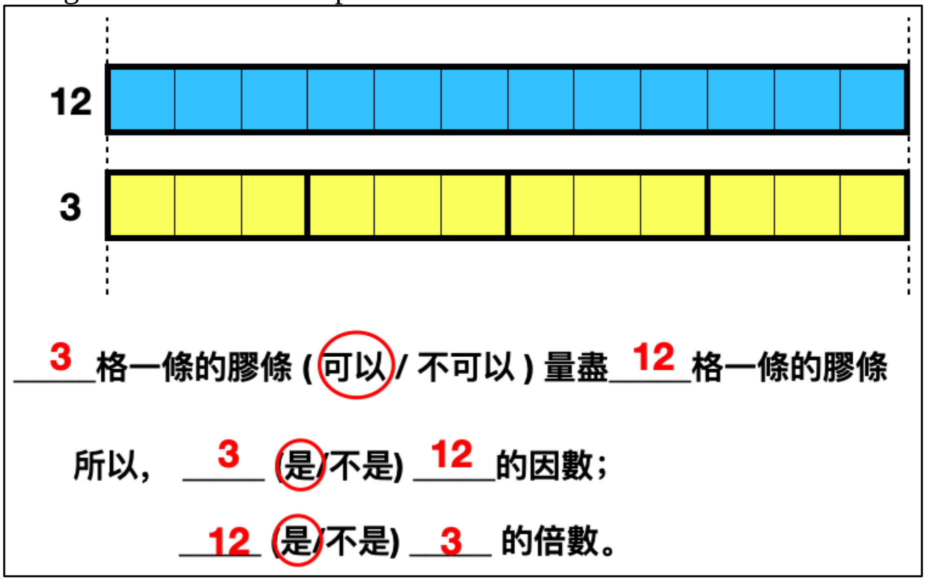
2. **Knowing the fundamental idea behind a learning topic.** For example, when talking about two-digit numbers, the subject advisor explained that the mathematics concept that needs to be carried out on the topic is the base-10 numeral system. The base-10 numeral system means (1) the value of a digit depends on its position, and (2) the place value of every column is ten times the place value of the column on its right. Learning activities on this topic should help students grasp the above two ideas. Recognising the fundamental idea behind the topic, teachers began to design activities that let students group items by ten and use the tens and ones they made to represent two-digit numbers.

3. **Visualise mathematics algorithms and concepts**. To help students gain a conceptual understanding of mathematics algorithms and concepts, teachers were eager to know how to make the content visible by students. An example of this is using number bars to explain the relationship between factors and multiples. The two number bars represent two integers (P and Q), with P and Q

as identical squares, respectively. If the corresponding number bar of P measures that of Q, then the integer P is a factor of integer Q; otherwise, P is not a factor of Q (see Figure 5.4 and Figure 5.5).

Figure 5.4

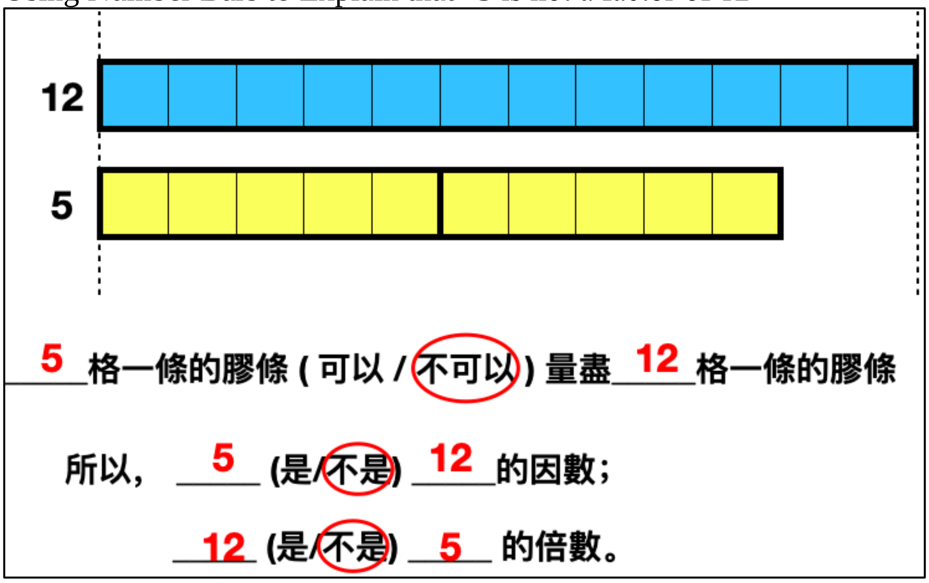
Using Number Bars to Explain that "3 is a factor of 12"



(In English: 3-square-long strips can measure a 12-square-long strip. Therefore, 3 is a factor of 12, and 12 is a multiple of 3.)

Figure 5.5

Using Number Bars to Explain that "5 is not a factor of 12"



(In English: 5-square-long strips cannot measure a 12-square-long strip. Therefore, 5 is not a factor of 12, and 12 is not a multiple of 5.)

4. **Identify the mathematical meaning behind a phenomenon**. This theme identifies the value of a phenomenon in developing mathematics, i.e., understanding the context in which learning takes place and the context through which mathematical knowledge could be developed. For example, a teacher asked a student to categorise "straight" and "bent" objects in her class, and she found the student did it well. However, when teaching "long" and "short," the teacher asked the student to categorise objects by "long" and "short," and the student failed to complete the task. Some teachers believed that the student had not yet become in command of the concept of long and short. However, the advisor and some teachers noticed that such a failure might also be caused by the design of the activity neglecting the difference between "straight and bent" and "long and short." In the concept of straight and bent, if a line is not straight, it is a curve. However, in the concept of long and short, they are relative concepts. An object is long only in relation to some other shorter objects. There is no absolute standard defining what is long, but there is an absolute standard defining straight.

For the KCS, teachers talked a lot about **what students are likely to think** and **what they will find confusing**. When designing activities, teachers always predict how their students will think or act in that activity and whether the task is easy or hard for them. Teachers also talked about the common difficulty of their students when learning a topic. For example, teachers said that students with moderate ID have difficulty counting out a requested number of objects; i.e., they cannot stop counting when they have reached the number requested.

Another theme of KCS is that **teachers differentiate the teaching of mathematics content based on their knowledge about learners' past learning experience and achievements**. Based on each student's knowledge background and ability, teachers assign different learning objectives and tasks to them. For example, a teacher needs to teach her students to solve a division problem. If a student has learned multiplication, then he or she should know how to use multiplication to calculate division. If a student has not learned multiplication, then he or she should learn how to obtain the answer by sharing or grouping counters.

To understand how the BE MATHS programme contributes to teachers' knowledge growth, the main source of the above knowledge was coded. Each case was coded as "mainly from teachers," "mainly from the subject advisor," or "developed by teacher and advisor together." Patterns can be found in the results shown in Table 5.2 (column contributor).

Table 5.2 shows that most SCK is contributed by subject advisors. As this knowledge needs a deep understanding of mathematics, the result is reasonable. Most KCS and PK are contributed by teachers. It is reasonable, as these two categories are the knowledge that grows in teaching. KCT is developed by teachers and advisors together.

The pattern shows that PK and KCS are the two kinds of knowledge that belong to teachers' expertise. However, SCK and KCT are the two categories in which teachers need external support in their practice, especially from subject experts.

How do SCK and KCT contribute to teachers' practice? How does the programme facilitate the growth of this knowledge? What moment has sparked the introduction or development of this knowledge? To gain an in-depth understanding of these questions, three cases were studied. Each of the cases describes a teacher's learning process along with the development of a *hypothetical teaching and learning trajectory*. When studying a *hypothetical teaching and learning trajectory*, many streams of knowledge are developed concurrently. To avoid information overload, only one stream of knowledge development is described in each case.

5.3 Teaching Directions to Students with Mild ID[1]

Karmen (pseudonym) graduated from a teacher training institute five years ago, with primary mathematics education being one of her major subjects. After graduation, she was appointed as a teacher in a school admitting students with mild ID.

5.3.1 Teaching Problems

Karmen and her colleagues chose "four cardinal directions" as the topic that they were going to study in the programme. The teaching problems that they had included were as follows:

1. Some of the students confused "upward direction" with "north direction";

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[1] I have published the findings of this case in the following paper:

Wang D., & Cheng K.M. (2020). Conducting design research for Mathematics in special education: A case of teaching directions to students with intellectual disabilities. *Hong Kong Journal of Special Education*. *22*(1), 17–28.

2. Students always made mistakes when they identified "north" and "south"; and

3. Most of the students in their school had difficulties finding the directions relative to a third party. For instance, given a school map, they could not complete the sentence: "The school entrance is at the ___ (direction) of the playground."

Karmen could explain the cause of the first problem, but she could not solve it. She said, "It may be a result of teaching via a map. Most maps point north at the top and south at the bottom, and students memorise the phrases we use when teaching ‘north is upwards, south is below, west is left, and east is right.’ I think these phrases confuse them." When talking about how to address the problem, Karmen did not give any suggestions. She said, "I know these phrases have caused problems, but I cannot find any alternative phrases to help them determine the directions."

For the second problem, Karmen guessed that students’ left and right confusion was the root cause:

> "We teach them ‘when you are facing east, and your back is towards the west. Your left is pointing towards north and your right points towards south’. If they have left and right confusion, it is hard for them to tell north and south correctly. Most ID students have left and right confusion, and we have tried to help them in different ways, but nothing is working as expected."

When talking about the solution to this problem, Karmen looked desperate. She said, “Teachers of ID students commonly accept that ID students have problems judging left and right. I don’t think I can teach them north and south if they cannot even judge left and right correctly.”

The careful examination of Karmen’s comments on the first two problems reveals the inadequacy in her understanding of cardinal directions. Cardinal directions and relative directions are two independent systems because relative directions are defined from an egocentric view, while cardinal directions are defined by referring to the North Pole. Karmen did not realise that defining cardinal directions by relative directions is inherently problematic. The statements that Karmen used to teach her students also showed a unified way to find cardinal directions. Sometimes, students were taught that their left was pointing west, and sometimes, they were taught that their back was towards the west.

Regarding the third problem, Karmen credited the learning difficulties with students’ autistic symptoms. "Students with autism have difficulty understanding another person’s point of view. When you asked an autistic student to find the directions relative to a third person, he/she would probably tell you the direction of himself/herself." It was the same regarding the second problem. Karmen thought there was little that teachers could do to enhance students’ performance because she believed that the problem was caused by students’ autistic symptoms, which were glaringly beyond teachers’ control.

5.3.2 Working on a Solution

In the school-based meeting, Karmen, the advisor, and three other teacher participants began their discussion with cardinal directions. The following questions evolved:

1. What are the cardinal directions? How do we define them? How is the cardinal direction system different from the relative direction system?

2. What is the difference between teaching cardinal directions in the subject of mathematics and the subject of geography or general study?
3. How can we help students find cardinal directions in a 3D space relative to themselves?
4. How can we help students find cardinal directions in a 2D plane (e.g., map, worksheet)? How could the students' experiences in a 3D space help?
5. How can students' skills in finding directions relative to themselves transfer to finding directions relative to a third person?

The advisor provided a brief analysis of the knowledge structure of cardinal directions with teacher participants. The following information was mainly provided by the advisor at the meeting:

- Left, right, forward, backward, up, and down are relative directions defined from an egocentric view. Different observers may face different orientations, which means that their relative directions may not match those of others and thus cause confusion in communication. Cardinal directions avoid this problem because they are defined with reference to the North Pole. North is defined as the direction pointing towards the North Pole, and south is defined as the opposite direction to north. East is the direction obtained after rotating north clockwise by one right angle, and west is the direction opposite of east. For the purpose of teaching students at these levels, the slight difference between the magnetic north and the geographic north is neglected.
- The cardinal direction to be determined in the first place requires knowledge of geography (e.g., using sunlight) or physics (e.g., using the Earth's magnetic field). Mathematics can be applied afterwards to determine the remaining three

directions. Specifically, when given one of the four directions, the other three can be determined according to the cyclical sequence "east, south, west, north" by successively rotating clockwise by one, two, and three right angles.

- Students have to say the following sentences aloud and turn together to find the cardinal directions:

  "When I face east, turn a right angle clockwise, then I face south;
  When I face south, turn a right angle clockwise, then I face west;
  When I face west, turn a right angle clockwise, then I face north;
  When I face north, turn a right angle clockwise, then I face east."

- Students are required to stand on a mat with a cross printed on it, with one arm pointing forward to match the direction they are facing. After being informed of the direction they are facing, they need to follow the sentences listed above to determine the remaining three directions by rotating their bodies and pointing their arms accordingly. The cross on the map serves as a reference for right angles.
- To help students transition from a first-person to a third-party perspective, a doll is given to students to rotate on a piece of paper with a cross on it. Students try to find the remaining three directions using the doll once the first direction is given.

The activities described above were introduced by the advisor and then further discussed with the participating teachers. The mats and dolls used in teaching were specially designed to help students with mathematics learning difficulties. Teaching experiment video clips of these activities were played to show how the activities were implemented in inclusive mainstream classrooms. All teachers agreed that the suggested activities had the potential to solve the teaching problems with additional adjustments to align the needs of students with mild ID.

Karmen thought the connection between finding directions in a 3D space and 2D space should be made more obvious to students. She suggested that teachers provide four labels with the words “north”, “south”, “east”, and “west” to students (see Figure 5.6 and Figure 5.7). When students find a direction in a 3D space, they need to put the corresponding label on the mat in the direction they are referring to. She said:

> “Whenever they want to find the directions, they need to read four sentences aloud, rotate their bodies or dolls accordingly, and label the directions. The series of actions establishes a correspondence between the direction pointer printed on a map and the cardinal directions in the horizontal plane. By labelling the directions on the mat, students will more easily understand that the cross is a sign pointing to the four directions. This learning experience is paving ways to learn direction signs on the map.”

Figure 5.6
Finding Directions from a Third-party Perspective

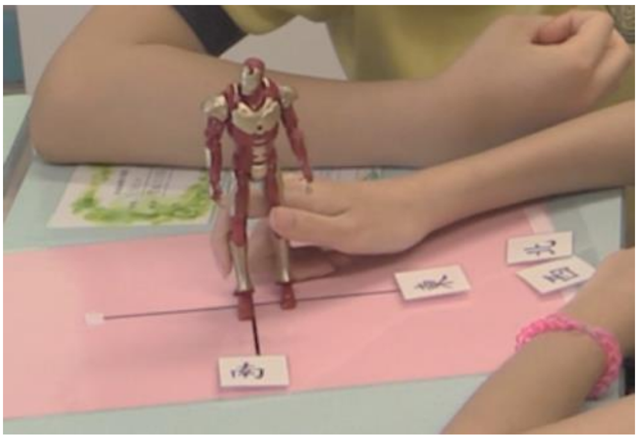

Figure 5.7
Finding Directions from a First-person Perspective

Karmen also revised another learning activity that guides students to understand the imperfect nature of the relative system. The first version of the activity asks students to sit face-to-face and put a yellow card to the right of a red card (see Figure 5.8); the teacher hopes that students sitting opposite will argue that the student is putting the card in the wrong place (as in his or her view, the card is put on to left of the red card). The teacher will then use this conflict to help students understand the imperfection of the relative direction system. Karmen thought the activity was suitable for students but still needed some revision. She explained:

> "First, students with intellectual disabilities are weak in determining right and left; second, some students may not be interested in checking whether the other student did it correctly."

Karmen revised the activity so that students are standing back-to-back in a hula hoop (see Figure 5.9), and then they are asked to go forward together. Then, the two students feel that the hula hoop limits them from going forward, as the other student is going in another direction. This revised version can avoid students' weaknesses of figuring right and left, which might present extra obstacles in the teaching process, and it creates a deeper impression among students that his or her classmate is going in a different direction.

Figure 5.8
Activity for Understanding the Imperfections of a Relative Direction System

Figure 5.9
Standing Back-to-back in a Hula Hoop

5.3.3 Teaching Experiment and Reflection

A teaching experiment including the above activities was carried out with a group of 13 students aged 12 to 14 with mild ID. It was found that more than ten students could determine the cardinal directions when one was given and that nobody pointed upwards when asked to point to north. More than ten students could find the directions on a map when given north was not pointing upward. Nine of them could tell an object's four directions from a third-person's perspective.

In the after-class review meeting, Karmen said students' comments on the shortcomings of the relative directions system exceeded her expectations. Before this class, she rarely asked students to comment on things because she thought that giving comments required critical thinking, which was beyond the ability of mild ID students. However, the students did it very well in the class, which surprised her.

In the sharing session of the project, Karmen said:

> "I think the design of this topic gives me a direction on how to break down the learning objects, and because of this, I can describe more about each student's learning progress.
>
> ... The teaching kits help my student learn how to find directions. My students' sense of space was weak. The mat helped my students establish the connections between pointing directions in three-dimensional space and a two-dimensional plane. I think my students have fully mastered the topic, and I believe they can transfer what they learn in this topic when they learn eight basic directions.
>
> ... I was impressed by the progressive learning trajectory designed for ID students learning cardinal directions. The transition of students from turning around to finding directions on a mat, then using a doll to represent themselves turning around on the paper, to finally, just imaging turning around the process in their head and determining directions on papers/maps is smooth and gradual. It reminds me that when I design my teaching, I need to consider the connections between activities.

> ... I was also impressed by asking students to read aloud the sentences about the direction. I think I had ignored the importance of learning math vocabulary and sentences before. In this experiment, I observed how the math sentences helped my students learn directions. After they repeat the sentences, again and again, these sentences just keep playing in their head when they are finding directions. The sentences lead their actions!"

When asked whether there was anything she wanted to revise in the teaching design, if she had the chance to teach it again, Karmen said she would help students strengthen their rotating clockwise concept at the beginning of this teaching unit. She found that when her students were pointing directions, they always needed a clock to remind them which way was clockwise in the teaching experiment.

5.4 Teaching Counting to Students with Moderate ID[2]

How Ken (pseudonym), who is a mathematics teacher in a moderate ID school, sought improvement in teaching counting to students with moderate ID is described. Ken is an experienced teacher who has taught in a moderate ID school for over 20 years.

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[2] I have published part of the findings of this case in the following paper:
Fung, C. I., & Wang, D. (2020). The pivotal role of knowledge structure and instructional design in the development of teachers teaching mathematics to students with special needs. *US-China Education Review A*, *10*(3), 113-125.

5.4.1 Teaching Problems

Ken found that some students in his school had studied counting for many years but could still not count over five. As counting is the skill that students must know before they can learn arithmetic, Ken was eager to look for advice.

In the first meeting, Ken and his colleagues presented what they have done to help students learn to count. Many teaching and learning materials were presented, including self-created songs of numbers, counting books with augmentative and alternative communication machines, and the school-based curriculum of counting based on variation theory. Ken is proud of what they have done to help students learn to count, but he has found that their methods only help their students count from 1 to 5; the methods do not work when the number is larger than 5.

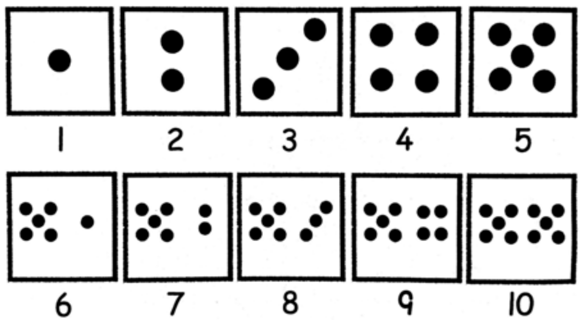
Ken said as follows:

> "[The mathematics teachers at our school] have discussed teaching problem related to counting many times. We have sought out and tried many resources, different approaches... yeah, there has been some progress (in student achievement) but not as much as we had expected. The teaching resources we developed proved to help some students learn the numbers up to five, but these techniques just didn't work when we continued teaching larger numbers."

Figure 5.10
Numbers and Image Patterns

(In English: “pencil” 1, “duck” 2, “ear” 3, “flag” 4, “hook” 5, “whistle” 6, “crutch” 7, “gourd” 8, “ballon” 9)

Figure 5.11
Numbers and Dice Patterns



Ken showed his teaching materials (see Figure 5.10 and Figure 5.11) to the advisor:

> “We have used images to help them. Each number has one fixed image. The dominant theme is connecting numbers and their amounts with a certain pattern. For example, the image of five is a quincunx (consisting of five points arranged in a cross, like the five-side of a die). When we teach them what five is, we ask them to arrange the objects in the pattern of quincunx and remember ‘whenever objects can arrange in this pattern, the number of objects is five.’”

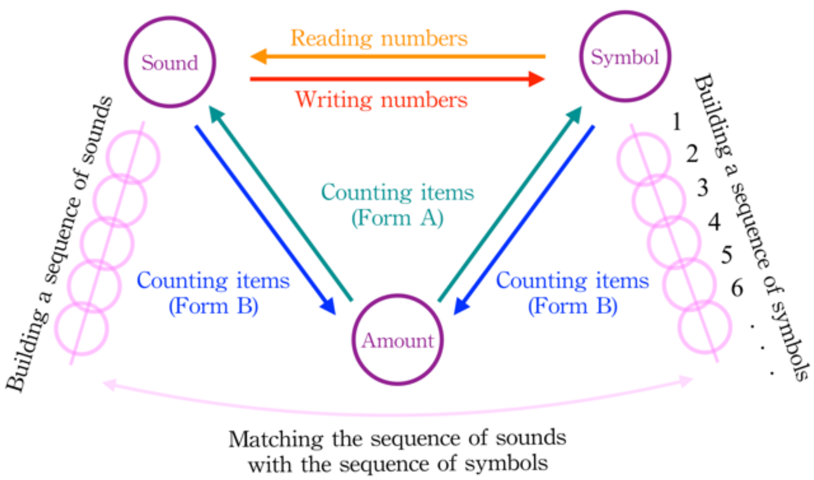
Ken said that they had applied variation theory in their design. Students would experience variations in the size, colour, and material of the objects while numbers and their patterns would stay the same. This multisensory technique was used to help students memorise numbers and their patterns.

Although there was no evidence that this approach indeed served to make somebody understand the numbers 1 through 5, including the related amounts behind these symbols, one thing was clear: the more numbers that students learn, the more geometric patterns that students have to remember. As there are unlimited numbers, students will eventually reach the limit of their memory capability, and then the approach will fail. Ken and his colleagues did not realise this problem. They observed that students experienced obstacles when learning numbers larger than five, but they had no idea about the reason.

5.4.2 Working on a Solution

The advisor used Figure 5.12 to introduce the knowledge structure of numbers and counting.

Figure 5.12
The Knowledge Structure of Counting



Going from 1 to 9, students are confronted with nine different symbols. Students need to learn the actual amount (in terms of the number of certain objects), the sound (as read aloud), and the symbol corresponding to each of them. This involves the following:

1. Reading numbers: Given the symbol of a number, finding the corresponding sound.
2. Writing numbers: Given the sound of a number, finding/writing the corresponding symbol.
3. Building number sequences: Reading number words or writing number symbols in a correct and consistent order.
4. Counting items (Form A): Given a set of items (amount), finding the corresponding number (its sound and/or symbol) for the amount.
5. Counting items (Form B): Given a number (its sound or symbol), picking up items equal to the amount of the corresponding number.

When asked to comment, Ken said:

> "I like it... Although I knew these components (sound, symbol, amount) before, it was not organised as well as this... This helps me to check what learning activity we are doing with my students and whether the learning experience we have provided is comprehensive."

To address the phenomenon in which Ken failed to help students learn numbers greater than 5, the subject advisor explained the abstraction principle of counting (Gelman & Gallistel, 1986); i.e., counting can be applied to any collection of objects, no matter what geometric patterns the objects are arranged. Using the geometric pattern to teach counting adds one unnecessary component to students' knowledge structure of numbers, which adds burden to both students' working memory and long-term memory.

After a detailed description of the number and counting, the advisor introduced a set of books to the teachers. The books were from the Louis Programme Training Centre (retrieved from http://www.lp.org.hk/e_index.htm) and are designed to help students with moderate or severe ID learn to count. In the books, repetitive counting tasks are arranged with progressive minor variations (see the following table). Each counting task involves pointing at objects on the left page while saying the number sequence and then pointing at the corresponding circles on the right page while repeating the number sequence. Page after page, the objects and the colour, size, position, and background of the objects are varied yet the circles on the right pages are unchanged, which guides student to link the circles that represent objects and sense the abstraction of quantity; no matter what objects they are pointing to or in what position, the quantity stays the same.

The books contain three levels of counting tasks. In the first level (see Figure 5.13 and Figure 5.14), there are no number symbols presented in the books. The figures on the left page are identical objects without any background image. The dots on the right page are arranged in the same position as the corresponding objects. To control the cognitive burden of the learner, number symbols are not included at this level. The goal of the activities is to help students (i) remember the sequence of sounds, (ii) match the sounds to the objects they see, and (iii) establish good habits in counting, which includes:

1. Counting with finger-pointing and eye-chasing.

2. Counting while saying the number sequence aloud.

3. Saying one number for one object.

Figure 5.13
Level 1

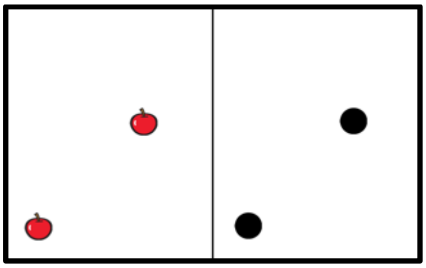
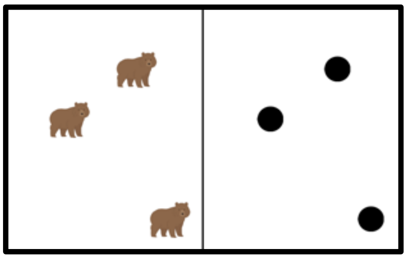


Figure 5.14
Level 1



Number symbols are added at the second level, and the circles are placed in line with number symbols tagged on them (Figure 5.15). The goal of this level is to help students remember the sequence of symbols and match them with the sequence of sounds. The background colour is added in the latter stage of the level to prepare students for counting objects with a more complex background in the next level (Figure 5.16).

Figure 5.15
Level 2

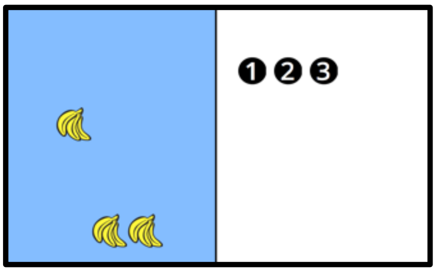
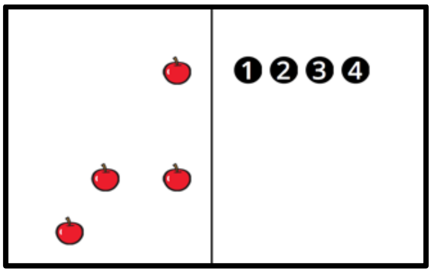


Figure 5.16
Level 2



At the last level, objects on the left page vary in size, direction, position, and distracting background (see Figure 5.17 and Figure 5.18). Going through the series of counting activities, the student will systematically establish counting skills.

Figure 5.17
Level 3

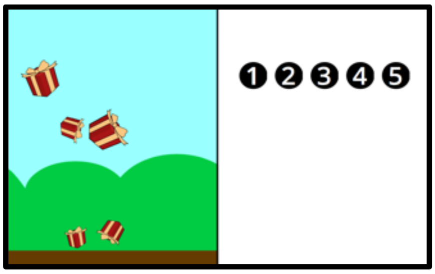
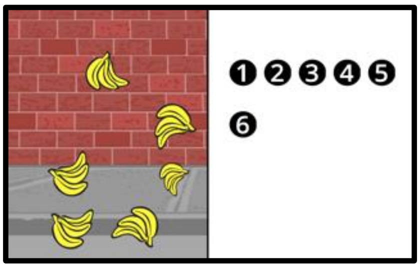


Figure 5.18
Counting Book Level 3



In comparison to the teaching materials that had been developed by Ken and his colleagues, the counting books not only help students match numbers to sound, symbol, and amount but also develop an understanding of the following principles of counting (Gelman, 1978):

1. *The one-to-one principle*: each object is counted only once;
2. *The stable order principle*: repeatedly using a list of tags in the same order to correspond to the objects to be counted;
3. *The cardinality principle*: the last number used to count the objects in a group represents the number of objects in the group;
4. *The abstraction principle*: any group of items can be counted in the same way; and
5. *The order irrelevance principle*: counting results are independent of the order in which objects are counted.

5.4.3 Teaching Experiment and Reflection

There were five students in Ken's class. Two of them could match number symbols with their sounds, and the other three could say the number sequence up to 10. One student had no language ability, but he could make sounds.

Before the teaching experiment, students were observed to exhibit many behavioural problems during their mathematics class. The students did not appear to be interested in Ken’s teaching. Ken had to keep stopping teaching to ask students to go back to their seats in an attempt to control them, and he had to ask them not to yell. Although there was a teaching assistant in the class, Ken looked exhausted from teaching and classroom management.

Teaching experiments are normally performed after the second or third meeting of the learning circle. However, Ken could not wait to test the counting books introduced by the subject advisor after the first meeting.

In the second meeting, Ken said he had already produced five sets of counting books and tested them with his students. He observed that when counting the apples on the left page, some students continued to count the page on the left (see Figure 5.19 and Figure 5.20).

Figure 5.19
Counting Task

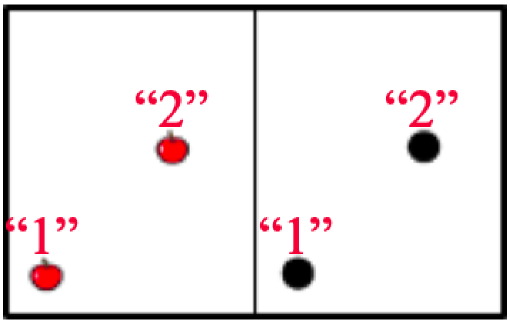
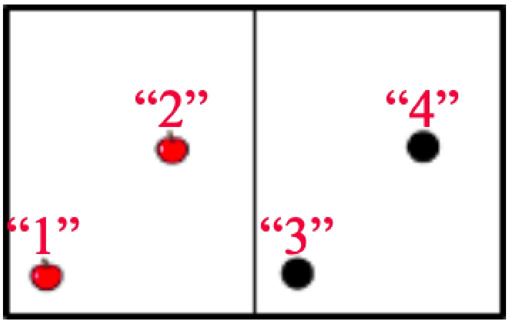


Figure 5.20
Students' Misinterpretation of Counting Task



Teachers in the learning circle pointed out that the problem was caused by printing the two pages on one piece of paper. They suggested that Ken print the left page and right page on two separate pages and then place them in two pockets of a clear book (see Figure 5.21) so that the students would easily understand that the left page and right page were two different counting tasks.

Figure 5.21
Revised Version of Counting Book

The teachers and subject advisors were invited to observe Ken's class one and half months after the teaching experiment began. Compared with the first observation of Ken's students before the teaching experiment, the most apparent change was that the disturbing behaviour of the students had been reduced. Students were sitting in their seats most of the time. Two students left their seats only because they had completed their learning tasks and were eager to report this to Ken. Another teacher observer commented that this was because the learning activities in this class were now aligned with the students' abilities, especially the students with a lower level of ability. All the observers agreed that the students could now follow the counting books well. To cater to the learning needs of the student without language ability, Ken gave the student a tablet computer to use a counting e-Book. When he pointed to the objects and the circles, the tablet would make the corresponding sounds. He encouraged the student to imitate the sounds from the tablet when counting. It was surprising that the student's sounds had slight differences when saying different numbers. This meant that the student was learning number sounds while doing the counting tasks from the books.

At the end of the programme, Ken was excited to share with other teachers of the programme that two of his students could successfully count to 7. Their performance remained stable even after a long period of school holiday.

He said:

> "I am impressed by the systematic learning steps designed for counting. I will continue to try the counting books and recommend them to my colleagues who have not participated in our meeting. It is good to cooperate with advisors and uncover a plan that is practical in class for moderate ID students."

5.5 Teaching Mental Objects to Students with Severe ID[3]

5.5.1 Teaching Problems

Alice (pseudonym) teaches students with severe ID. For the majority of these students, their intellectual disability is accompanied by physical disabilities and severe language impairments. Alice described the plight of her teaching as follows:

> "… Sometimes, I give them an apple and said, 'This is one'. Sometimes, I give them some heavy things and said, 'This is heavy'... but do the students understand what I am saying?"

Another teacher, Kaley (pseudonym), also had similar experience in her teaching:

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[3] I have published the findings of this case in the following paper:
Fung, C. I., & Wang, D. (2019). Teaching mathematics to students with intellectual disability: What support do teachers need? *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 4644-4651). Utrecht, Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University.

> "For example, I would like to teach them the concept of 'being less'. During the activity, 'being less' will be the different one among the choices. If you directly point to the group with fewer objects and tell the student 'this is being less' [without drawing their attention to the difference in amount], they will mistakenly name the objects they see 'being less'. Actually, they do not get what you mean, and they haven't even compared them [the quantity]."

The problems described by the two teachers were concerned with the teaching of mental objects to a learner with severe ID, severe language disability, and physical disabilities. Number, length, circle, square, weight, straight line, etc. are mental objects (Freudenthal, 1980), also called primary concepts (Skemp, 2012), which are commonly encountered in mathematics lessons. It is a great challenge for people to explain what they mean because they are the most fundamental ideas of mathematics that are generally acquired at an early age when people lack the ability to analyse them (Skemp, 2012). Although definitions of these concepts can be found in some dictionaries, how can teachers convey those messages to students with severe ID? How can students with basically no knowledge of language understand these linguistic explanations? How can the teachers check if students indeed acquire what they intend to communicate? When verbal communication fails, teaching needs to draw students' attention to mental objects without language. This is the common challenge faced by teachers of students with severe ID.

5.5.2 Working on a Solution

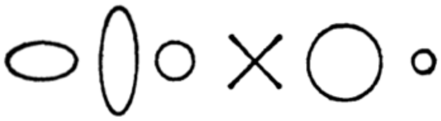
"Primary concepts" are derived from one's sensory and motor experiences of the outside world. The attainment of these concepts requires the formation of a number of experiences with something in common. Therefore, a primary concept could be seen as the "end-product of becoming aware of similarities among one's experiences" (Skemp, 2012, p. 11). To teach students, a primary concept is to provide a collection of examples or experiences with common properties that can form the concept.

As concepts are the end-product of the formation of the similarities of experience, the attainment of the concept does not necessarily require a linguistic description of the concept. As Freudenthal (1980) observed, "concepts are defined operationally within a system of experience and contextually within a written description of this experience. It can be shown by many examples from the sciences that neither an explicit definition nor a name are needed for the attainment of a concept (p. 109)." Although using languages, such as name and definition, can speed up the formation of a concept, they are not a required component.

Although we know that experience is the essential element that forms primary concepts, how can students with very limited abstraction ability recognise the similarity among the experiences? According to Skemp (2012, p. 11), "contrast" is one factor that facilitates the process of abstraction of experience. From Figure 5.22, with the contrast of "X", the similarity of the five variously shaped "O's" stands out and is more likely to be remembered and abstracted.

Figure 5.22

A Single "X" Stands Out Perceptually from the Five Variously Shaped "O's"



Quoted from *The Psychology of Learning Mathematics* (p. 11), by R. R. Skemp, 2012.

Therefore, teaching should provide students with many examples and non-examples. The teachers conveyed a class of objects that shared certain properties without ever explicitly going into those properties in detail. Based on this understanding, a framework called "Pick the Odd Out (POO)" was suggested by the subject advisor for designing a series of multiple-choice activities under which a student could progressively develop a sense of a mental object without relying too much on verbal communication. Although the teacher is free to employ any verbal explanation during its execution, the main thrust that drives students to the target mental object comes from the teacher's confirmation of the correctness of a student's choice, which may well be done using body language or facial expressions.

Various intermediate stages of the POO should be portrayed. These stages should encompass sufficient variation across examples and non-examples of the mental object to be conveyed. First, there should be a key concept, or a focus mental object, that appears in the series of multiple-choice activities. Going through the series of multiple-choice activities, the student will systematically visualise or experience these variations. Each multiple-choice item includes four or five choices, with at least one choice corresponding to the focus mental object and at least one choice corresponding to the other. The student should indicate (by pointing or other means) the odd one out. Through systematically varying these choices, the student's attention is drawn to the focus mental object. Timely feedback from the teacher serves to shape how the mental object is developed and hence is an indispensable component of the activities.

There are four stages of the POO. In Stage 1, each item comprises exactly one choice that corresponds to the focus mental object (straight line in this case), while the remaining are identical choices of non-example in identical orientation (Figure 5.23). In Stage 2, each item comprises exactly one choice that does not correspond to the focus mental object, while the remaining are identical choices of the focus mental object, in identical orientation (Figure 5.24). Variations of the forms of the mental object and the otherwise across items should strive for comprehensive coverage as far as possible (see Figure 5.25 and Figure 5.26). In Stage 3, each item comprises exactly one choice that does not correspond to the focus mental object, while the remaining choices correspond to the focus mental object in different forms (Figure 5.27). Up to this point, the student should have seen or experienced many examples and non-examples of the focus mental object. The teacher can now reveal relevant terminology in written or oral form. In Stage 4, no specific restriction is imposed on the choices, and the student is required to pick out the unique choice that corresponds to the focus mental object by its name (Figure 5.28). In other words, the student tackles multiple-choice questions commonly found in regular mathematics classes only in Stage 4. All the previous activities are designed to fill the gap before the student is capable of handling this step.

Figure 5.23
POO Stage 1

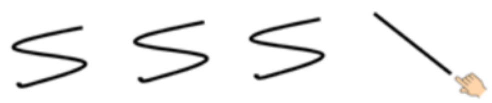


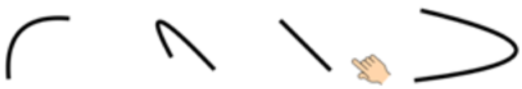
Figure 5.24
POO Stage 2

Figure 5.25
Variations of POO Stage 1

Figure 5.26
Variations of POO Stage 2

Figure 5.27
POO Stage 3

Figure 5.28
POO Stage 4



In principle, this framework can be applied to teach a variety of mathematical mental objects or informal notions, such as straight lines, circles, emptiness, being long, inner parts, being heavy, and many.

5.5.3 Teaching Experiment and Reflection

Alice conducted the first teaching experiment of POO. The results indicated that there should be a preparatory stage (see Figure 5.29 and Figure 5.30) to confirm that the student can match objects or pictures at hand with a group of identical objects or pictures not necessarily in the same orientation. Tasks are divided into four parts, as indicated in Figure 5.29 to Figure 5.32. In each case, the student should put the object or picture at hand into the same group. After the successful completion of these activities, we can reasonably assume that the student has developed a sense of grouping the same (the meaning of which will be uncovered progressively) things together.

Figure 5.29
Identical Objects in Identical Orientations

Figure 5.30
Identical Objects in Different Orientations

Figure 5.31
Groups of Identical Objects in Identical Orientations

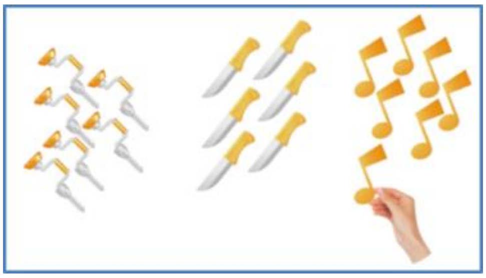


Figure 5.32
Groups of Identical Objects in Different Orientations

After inserting the preparatory stage, the second round of the teaching experiment was conducted with four students (S1, S2, S3, and S4) with severe ID who had little communication ability. One student (S4) has autism spectrum disorder and always exhibits restricted and repetitive behaviours. The learning pace of the students was slow, but progress was still observable, as shown in Table 5.4.

Table 5.4
Learning Performance of Severe ID Students Across Video Recorded Lessons

Student	1st lesson	2nd lesson (after 6 wks.)
S1	Able to choose the straight line	Able to classify straight lines and curves
S2	Cannot choose the straight line from four choices	Able to classify straight lines and curves
S3	Cannot classify straight lines and curves from four choices	Able to classify straight lines and curves
S4	Cannot give teacher an identical object	Able to put two identical objects into a group

Students' learning progress was also observed by the teachers who attended the teaching experiments as observers. They all agreed that the framework helped their students learn mental objects.

Alice found that the framework provided a way for her to communicate with severe ID students. She had never thought that POO could be an effective starting point to learn a mental object. She also found that POO not only assisted her students' learning but also enhanced her professional competence in teaching students with severe ID.

Interviewer: What changes have you had after participating in this teaching experiment?

Alice: Before this teaching experiment, I didn't know what content I should teach [to students with severe ID]. Now, when I teach mathematics, I have something to follow. I can refer to the framework. There are many levels and steps, so I can evaluate what level a student has achieved and know what learning activities should be assigned to the child at the next level. This makes lesson planning easier because the framework enables me to plan my teaching for each individual student.

Interviewer: What changes do your students exhibit?

Alice: One student could pick the odd one out only when it differed from the others significantly; now she can do it with even a slight difference. I found that when they are familiar with the learning activity, they can make progress much faster. It is good for them to learn different topics under one progressive learning framework. Before this teaching experiment, I included a variety of learning activities in my class because I believed doing so would make for a fruitful lesson. However, from the perspective of students with severe ID, this is not the case. They cannot handle a frequent change of content and learning methods and consequently, they make little progress.

Interviewer: Did your students perform up to your expectation?

Alice: More than half of the students performed better than I expected.

Interviewer: What was your expectation before?

> Alice: Before? I didn't expect much. I just gave them some manipulatives related to mathematics and hoped they could explore by themselves. I was satisfied if they could touch and play with them. However, now, I expect more. I hope they can formulate some mathematical ideas out of the activity instead of just playing.

5.6 Summary of the Three Cases

5.6.1 Need Supports of SCK

Paralleling the findings in Section 5.2, pedagogical knowledge and knowledge of content and students were the two kinds of knowledge that were mainly contributed by teachers, while specialised content knowledge was mainly contributed by subject advisors, and knowledge of content and teaching was developed by subject experts and teachers together.

The three cases further strengthen the conclusion in Chapter 5.2 that teachers in the learning circles have substantial PK and KCS in regard to teaching ID students. They are sensitive to students' needs when selecting and designing tools and other learning materials, for example, selecting larger toys in consideration of students' weak finger muscles and applying structured teaching strategies for students with autism disorder. Teachers also have a good command of knowledge of content and students, where they can describe students' common errors and difficulties related to learning specific content. They understand their cognitive needs, break tasks into small steps, use manipulatives supporting learning, and design various activities to meet their SENs.

However, all three cases showed that having PK and KCS was not enough to solve the related teaching problems. The expertise needed in mathematics teaching for ID students goes beyond these two kinds of knowledge.

As observed in the three cases, there is a learning process immediately after a teacher is confronted with a subject expert's deliberation and explanation of the content for teaching, for example, the attainment process of mental objects, the difference between relative direction system and cardinal direction system, the components of counting, and the abstract principle of numbers. With a deeper and more structured understanding of the content for teaching, teachers have a clear direction for their teaching. They can further modify the instructional design into a version that can engage their ID students. This result suggests that the SCK introduced by the subject experts is the key to kicking off teachers' development of KCT and KCS.

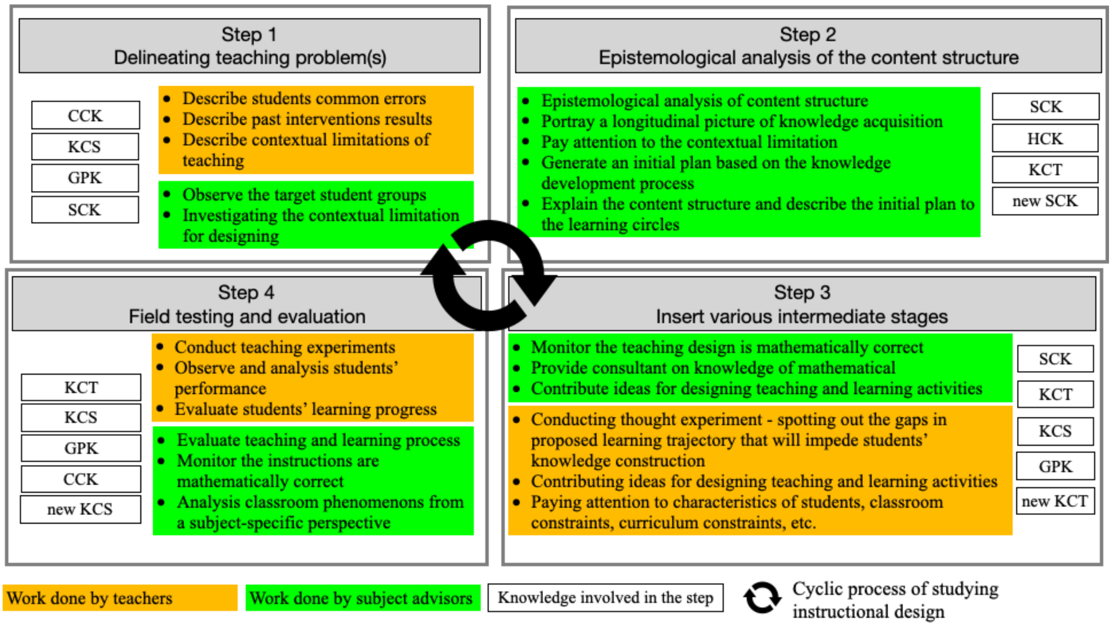
To develop curriculum materials and instructional designs for ID students, the following four steps were identified in the three cases (see Figure 5.33):

1. Delineating the teaching problems that teachers are confronted with, conducting observations on the target student groups, and understanding whether there are any constraints need to be considered when designing the teaching and learning trajectories;
2. Carrying out an epistemological analysis of the content structure to design a skeleton plan of the hypothetical teaching and learning trajectory.
3. Inserting various intermediate stages into the hypothetical learning trajectory, paying attention to factors such as student characteristics, classroom constraints, and curriculum requirements.

4. Field-testing the plan and collecting empirical evidence for further improvements of the design.

Figure 5.33

The Process of Improving Mathematics Teaching for ID Students



Based on the examination of the knowledge involved in the four steps, the second step needed SCK the most and was where teachers needed to be supported the most. If that step is done well, teachers will have a clear direction for their teaching and can further modify the instructional design into a version that can possibly engage their ID students.

5.6.2 Relationship Between Domains of Knowledge for Teaching

It is interesting to find that in all three cases, there was a causal relationship present between KCS, SCK, and KCT.

In all three cases, the subject advisor's analysis of SCK made a breakthrough to solve students' learning problems and facilitated the growth of KCT. However, teachers' sharing about their KCS in a specific learning content stimulated subject advisors to study and analyse the learning content and find the cause of the teaching problem from a subject-specific perspective. Without teachers' contribution in pointing out the learning difficulties and teaching problems, the SCK will never have come on the stage or played a role in solving learning problems.

Teachers' contributions to KCS and PK turns ideas from advisors into practical and effective learning activities. In the case of "direction", the teaching plans provided by the advisor considered the connection of direction in 3D space and direction presented in a 2D plane. However, the results of the first teaching experiments showed that students did not grasp the connection as expected. The teachers' idea to use labels to make the connection more significant for students led to successful teaching. In this case, the contribution of PK to the development of KCT is evident. In the case of teaching mental objects, the teachers, based on their KCS, suggested one previous level needed by their students. This suggestion contributed to the development of an instructional design for teaching mental objects. From the above two cases, it is found that the development of KCT also needs support from KCS and PK.

The development of SCK and KCT was facilitated by the programme structures, which provided teacher opportunities to discuss mathematics content with subject advisors. This showed that collaboration between subject specialists and special education teachers could promote successful mathematics teaching in special education. As observed in the case of teaching cardinal directions, the instruction of cardinal direction designed by teachers alone neglected its knowledge structure and was problematic in teaching. The learning activities suggested by the subject experts also neglected students' special learning needs and were difficult to implement. Having a combination of special education teachers and subject experts in the research team ensures that the instructional designs developed are mathematically sound on the one hand and practically feasible to students with cognitive limitations on the other hand.

Chapter 6 Conclusion, Discussion, and Implications

6.1 Summary of Quantitative and Qualitative Findings

Four research questions guided this mixed-method study. Data were collected from teachers' questionnaire responses, observations of students' mathematics learning engagement time, teacher interviews, observations of BE MATHS programme activities, and reviews of the resources developed in the BE MATHS programme. Each research question frames the findings of the data.

6.1.1 RQ1: Teachers' Views on Their Professional Development Experiences

The results based on the quantitative data show that compared to the in-school professional development activities generally available for teachers, the BE MATHS programme scored significantly higher on all CTPD subscales. The scores indicate that the BE MATHS programme activities **encourage more collective participation, are more content-focused, are more coherent with teachers' needs, and involve more active learning both inside and outside the classroom**.

Qualitative data also support the above quantitative results. First, **collective participation** for the BE MATHS group was reflected in the collaborations between teachers from different schools and the partnerships formed between teachers and subject advisors. Teachers of different levels and from different schools, along with experts from various schools and universities, approached teaching problems together. They also shared their experiences and knowledge about teaching and learning and exchanged their ideas.

Second, the aim of the programme activities to provide instructional designs for solving teaching problems determined a **content focus** on specific topics in mathematics. During the development process of instructional designs, subject advisors analysed teaching problems from the subject-specific perspective of mathematics, explained relevant subject content to teachers, and designed teaching and learning activities together with teachers. Therefore, content knowledge for teaching was frequently studied and remained the focus of attention and discussion.

Third, all interviewed teachers mentioned that the materials developed through the programme were essential and urgently needed by them. They commented that the materials provided learning trajectories that they failed to find before they were coupled with directions for their teaching. It was also observed that all programme activities were centred on solving teaching problems by developing teaching and learning trajectories for ID students. As teachers felt that solutions were found for such problems, they readily perceived the programme to be **coherent with their needs**.

Last, the teachers were regarded as researchers themselves who needed to contribute their knowledge and experience in developing learning trajectories. They participated in thought and teaching experiments, thereby showing an **active role both inside and outside classrooms**. Teachers were actively involved throughout the entire development process of instructional design. They decided on the teaching problems to investigate during the programme; shared their challenges and experiences in teaching; contributed their expertise in students' learning preferences and pedagogical knowledge; discussed teaching aids, worksheets, and learning activities with the subject advisors and other teachers; conducted teaching experiments and evaluated their results; and reflected on their teaching and revised their teaching designs.

In conclusion, all five attributes for effective professional development experiences as measured by the CTPD were found to be interwoven into the BE MATHS programme activities concerning the development of teaching and learning trajectories.

6.1.2 RQ2: Teachers' Changes in Teaching Approaches Following Programme Participation

The changes that teachers exhibited following their participation in the BE MATHS programme occurred in the two aspects of personal mathematics teaching efficacy (PMTE) and mathematics teaching outcome expectancy (MTOE). The quantitative data show that the programme positively impacted those participating teachers with pretest PMTE scores that were above the average for all teachers. Furthermore, the study found that the programme significantly influenced participating teachers' MTOE compared to teachers who did not participate. Likewise, the qualitative data also show that certain teachers mentioned that they raised their expectations about their students.

The qualitative data provided substantial evidence on the improvement of teachers' expertise in mathematics teaching ID students. First, by studying hypothetical teaching and learning trajectories for ID students, teachers were able to identify the missing parts of their knowledge that prevented them from providing students with a complete or satisfactory learning experience. Second, the trajectories were also able provide teachers with a detailed picture of how ID students learn. This helps teachers design or arrange teaching and learning activities more systematically.

It was observed that teachers' knowledge of content and teaching (KCT), specialised content knowledge (SCK), and knowledge of content and students (KCS) grew during the design of hypothetical teaching and learning trajectories for ID students. Concerning KCT, teachers knew more about the following:

(1) Learning activities and tools that make specific mathematics content comprehensive to ID students;
(2) Tools and aids in assisting students in completing mathematical tasks;
(3) The level of guidance in helping students complete a mathematical task;
(4) The vocabularies and sentences that students need to learn; and
(5) The process for abstracting mathematics content from manipulative.

Concerning SCK, teachers improved their understanding about the following:

(1) The knowledge structure of a mathematics content;
(2) The fundamental idea behind a learning topic;
(3) The ways that visualise mathematics algorithms and concepts; and
(4) The mathematical meanings behind a phenomenon.

The growth of teachers' KCS showed that teachers knew more about students' thinking and misconceptions in regard to certain mathematics content.

6.1.3 RQ3: Students' Changes in Engagement with Mathematics Learning

The BE MATHS programme can help teachers make their students who rarely engage in mathematics classes become more **actively engaged** in mathematics learning.

However, for students who were already engaged in mathematics classes before their teachers participated in the programme, the programme did not have a significant impact on their learning engagement. Nevertheless, teacher participation in the programme resulted in decreases in students' **active nonengagement time**. The quantitative findings and the qualitative findings are convergent. The three cases described in Sections 5.3.3, 5.4.3, and 5.5.3 show that most students were observed to have better academic achievement in the teaching experiments.

6.1.4 RQ4: The Contributing Factors of the Changes

Facilitating professional development via a design research approach. According to the interview data, all the teachers mentioned that they had learned much from designing hypothetical teaching and learning trajectories for ID students. Before participating in the BE MATHS programme, certain parts of certain learning trajectories functioned as their blind spots, which the programme helped identify, which improved their teaching. The teachers' comments on the teaching and learning trajectories developed through the programme indicated that they used the trajectories as reference frameworks when making didactical decisions.

Providing subject-specific support. SCK introduced by the subject experts is the key to kicking off teachers' development of KCT and KCS. The SCK comes from subject experts' epistemological analysis of the content structure. It helps teachers identify the cause of teaching problems and portrays a longitudinal picture of mathematics knowledge acquisition. Although teachers rarely mentioned SCK in their interviews, the in-depth analysis of the three cases (see Sections 5.4, 5.5 and 5.6) shows that SCK introduced by the subject experts was a required component for developing instructional designs to solve various mathematics teaching problems and thus facilitate new understanding about ID students' mathematics learning.

Both special education teachers and subject experts were involved in the research team. Collaboration between subject specialists and special education teachers could promote successful mathematics teaching in special education. Although subject experts' SCK made breakthroughs in helping to solve teaching problems, without teachers' contributions in regard to pointing out the learning difficulties and teaching problems, the SCK would never have become so fully developed. Furthermore, teachers' PK and KCS turned ideas from advisors into practical and effective learning activities. A combination of special education teachers and subject experts in the research team ensures that the teaching and learning trajectories developed are mathematically sound and practically feasible for students with cognitive limitations.

6.2 Conducting Design Research for Mathematics in Special Education

6.2.1 Discussion on Design Research

Design research, which teachers and subject advisors in the BE MATHS programme engage in, is a kind of research that studies the educational field via an *engineering* approach. Burkhardt and Schoenfeld (2003) discussed the differences between engineering approaches and two other kinds of approaches, called *humanities* and *science.* They pointed out that the products generated by *humanities* or *science* approaches, which are commonly published in research journal papers, books, and conferences, are the inventions and generations of ideas, insights about a phenomenon, problems identified, and suggested possibilities. These results alone do not generate practical solutions that can be directly implemented. In contrast, *engineering* approach research is directly concerned with practical impacts, which improves the educational field by "designing and systematically developing high-quality solutions to practical problems" (p. 5). Therefore, *engineering* approach research can provide strong linkages between the academic field and practice. However, Burkhardt and Schoenfeld stated that such 'engineering research', which is quite common in other applied fields such as medicine, engineering, or electronics, is largely missing and undervalued in the educational research community.

The findings of this study provide evidence on how design research could improve mathematics teaching in special education settings. As mentioned in the introduction of the study (see Section 1.1), teachers in special schools are suffering an immense shortage of teaching resources. The design research assists the field in the most direct way, i.e., by providing teachers with carefully designed teaching and learning trajectories and learning materials that are tailored to students in special education.

Apart from solving practical teaching problems and developing valuable teaching and learning resources, design research on students' learning trajectories also has the potential to support teachers' understanding of student thinking and improve teaching and learning (Edgington, 2014; Heuvel-Panhuizen, 2001; Mojica, 2010; Wilson et al., 2013). This study confirms these findings through observations and interviews with teachers. As summarised in Section 6.1.2, teachers learn much from the study of teaching and learning trajectories.

The analysis of teachers' knowledge growth during the design process in the current study provides an explanation for teachers' professional development in studying teaching and learning trajectories. As shown in Figure 5.33, to solve a teaching problem, the research team undertakes an epistemological analysis of the relevant contents and structures. If this step is done well, it will help teachers on the research team to better structure their knowledge of mathematics and thus gain a deeper understanding of what they teach. In the next step, the research team focuses on designing details of the teaching based on their improved understanding of the mathematics content. New knowledge of content and teaching is developed during the design process. During the field tests, the proposed teaching and learning trajectories are examined and validated. Students' performances and classroom phenomena are analysed and discussed by the team, which facilitates teachers' understanding of students' mathematical thinking.

Although teaching and learning trajectories have the potential to enhance teachers' mathematics teaching, Empson (2011) pointed out that there are two pitfalls related to providing learning trajectories to a teacher. First, a teaching and learning trajectory represents learning as progressive sequences, which could lead teachers to direct students in a sequential way and thereby lose the diversity of mathematics understanding. Second, students' mathematical thinking and understanding have many possibilities that may not follow a predictable trajectory over time.

The presupposition behind these pitfalls is that teachers are regarded as the recipient of developed learning trajectories, which is not the case in the BE MATHS programme. Teachers are actively participating in the development process of teaching and learning trajectories. Their expertise in teaching is recognised as important knowledge that supports the development and refinement of instructional design. When teachers are trained and regarded as partners in the study of instructional designs, their flexible use of the materials is facilitated and stimulated (Wittmann, 2021). From the cases illustrated in Chapter 5, we can see the dynamic nature of a teaching and learning trajectory; designing, testing, and revising teaching and learning trajectories could help teachers and researchers better understand students' different thinking and understanding of mathematics. Furthermore, this better understanding of students' thinking and the content for teaching will continue to inspire new rounds of designing and revising the trajectory. This means that a teaching and learning trajectory will never be a mechanical procedure; rather, it will provide opportunities for diverse trajectories in mathematics learning to evolve.

The growth of the professional expertise of teachers and subject experts during the process of creating and refining the instructional design reflects Wittmann's notion of mathematics education as a systemic-evolutionary design science (Wittmann, 1995, 2001, 2021). Knowledge is not a result of transmission from teacher educator to teacher; rather, it is conceived as a productive achievement between teachers and the teacher educator. This is based on the belief of "self-organising powers inside the system." In Wittmann's words:

> "Recommendations and instructions from outside which do not fit into the internal processes of the system are, at best, useless. If, in addition, minute control is exerted from the outside, the development of spontaneous powers inside the system is suppressed, which undermines its efficiency. A system without a proper infrastructure is not able to interact adequately with a complex environment: variety can only be absorbed by variety." (Wittmann, 2001, p. 8)

The view of systemic-evolutionary design science rests upon the understanding of the complex nature of mathematics education. Too many uncontrollable factors and parameters prohibit the results of a teaching experiment in one setting from being unconditionally transferred to other settings. St. Clair (2005) described the result of education research as “empirical heuristics”. He said:

> “The empirical heuristic function suggests a more interesting task for the researcher, that of making sense of that finding in context and developing it into a tool for the thinking of other teachers. Even if the findings never leave the experimental site (setting A), the research will have done its job if the teachers involved in the teacher research within this approach, and indeed this form of research can be highly valued as the most direct possible application of the heuristic because the teacher reflects and learns during the research process itself. Empirical heuristics do not derive their value from correspondence to truth but form their potential to assist reflection, a function unrelated to truth or falsehood. If these limitations are recognised, empirical heuristics can be an effective and insightful way of using experience from setting A to inform the work of educators in setting B.” (p. 449)

Due to the complex nature of teaching, improving mathematics education by providing teachers with a ready-made instructional design is destined to fail. The value of designing instructional designs and conducting experiments on them is to involve teachers extensively in the research process, through which teachers reflect, learn, and then transfer their learning experience into other settings (St. Clair, 2005).

6.2.2 Implications

Despite the growing call for staff development in inclusive education and special education, actual professional development attempts by studying instructional designs are still limited. Although there has been an increased interest in using design research to promote teachers' professional development, for example, Japan's approach of "lesson study", an extensive interest in the approach among the field of inclusive education or special education has not yet been established.

The results of this study are useful for informing the community of teacher educators and policymakers about the potential of the design research approach in fostering the development of teachers' profession in caring for students with SENs. In addition, the encouraging results relating to teachers' change of mathematics teaching efficacy, growth of knowledge, and the increase in students' academic engaged time support the view that the design research approach is a promising alternative to existing practices of teacher training for promoting education for all.

More importantly, the findings of this study can serve as a starting point where further development and improvement in terms of the design of teacher professional development programmes can be discussed. In addition to using a design research approach to develop curriculum materials and teaching resources, teacher educators can apply design research as a tool to develop teachers' professions and capabilities in teaching. The framework for improving mathematics teaching in special education (see Figure 5.33) could be applied in other teacher training programmes, such as engaging preservice and in-service teachers in cycles of thought experiments and teaching experiments and developing instructional designs with subject experts, by regarding teachers as researchers in educational study.

The analyses of teachers' knowledge growth during the design process of teaching and learning trajectories contribute new insights into how domains of knowledge support teachers as they engage in design research to improve mathematics teaching.

6.3 SCK for Teaching Mathematics to ID students

6.3.1 Discussion on the Need for SCK

When talking about teaching methods that teachers can employ to support ID students, the following two strategies could be found in almost every resource book for teachers:

- Breaking down learning task into small steps; and
- Providing multi-sensory learning tasks.

However, applying these strategies in mathematics teaching is harder than it sounds. From the findings of this study, we know that without an understanding of the structure of the specific content, teachers do not know how to break the learning process of the concept into smaller steps (the case of teaching mental objects and the case of teaching cardinal directions); teachers provide multisensory learning tasks at a superficial level, thus increasing the unnecessary burden in learning (the case of teaching counting). These findings reconfirm Polya's conviction that (1963) "the psychology of learning may give us interesting hints, but it cannot [*sic*] pretend to pass ultimate judgement upon problems of teaching (p. 605)" A teacher's ability to analyse the problems of teaching mathematics, design teaching, and learning trajectories caters to ID students' SENs in mathematics learning and rests upon his or her understanding of the mathematics content for teaching. Therefore, applying general teaching strategies for catering to students' SEN must be based on a thorough understanding of the underlying subject matter.

Although the mathematics covered in special education is mostly elementary, the mathematical content knowledge needed in teaching is not elementary at all. It is common for a teacher to know the subject content but not to know about the cognitive construction process of that content. Using counting as an example, counting is near instinct for people with normal intelligence, i.e., just reading out a string of numbers. To a student with moderate or severe ID who is learning to count for the first time, they must learn to coordinate their eye movements with their finger pointing, to match a sound with each finger pointing, and how to do these things simultaneously. These skills are simple to people with normal intelligence but a masterstroke of genius for a student with moderate to severe ID. It is difficult for intelligent people to recognise the detailed smaller learning steps within the process. They have no experience or knowledge about how this simple concept activity actually consists of several or even many smaller components. If a teacher cannot see the components and the structure of a "simple" piece of mathematical content, then he or she may not know where teaching should start or how he or she can help.

In special education, *task analysis* has been widely used and is regarded as an important instructional tool for educators (Carter & Kemp, 1996). A core element of task analysis in special education is identifying and sequencing the components of a task. It involves breaking a skill down into smaller, more manageable steps. The number of steps in a task analysis will depend on how complicated the skill is that the student is learning. This approach has been used to teach a variety of skills, including but not limited to academic, behavioural, social and communication skills.

However, the findings of the current study suggest that the epistemological analysis of content knowledge and its structure need to be done before applying any task analysis strategy in mathematics teaching. This approach provides teachers with a direction in identifying the learning components and sequencing the learning steps in a constructive way. In the current study, some teachers applied task analysis strategy without having a good epistemological analysis of the mathematics content and thus caused teaching problems. For example, in the case of teaching counting to students with moderate ID (Section 5.4.1), the teacher, Ken, broke counting into three smaller steps: recognising the fixed pattern of a number, arranging the objects in the pattern, and recognising the number of objects. However, because of violating abstraction principal of number, the instructional design based on Ken's analysis only helped students learn up to five and failed to help them learn more numbers.

The epistemological analysis of content for teaching is different from the work of task analysis in the following aspects:

(1) The content for task analysis will not extend beyond the content of concern, while the epistemological analysis of the content is concerned with the longitudinal picture of the knowledge construction process of the content from which the knowledge comes.

(2) The direction for task analysis is top-down and therefore breaks down the content concerned into smaller components. In contrast, the direction for epistemological analysis is down-top and thus identifies the phenomenon of the content and analyses the process that organises the phenomenon into more abstract mathematics ideas.

When the learning steps are identified through breaking down mathematics content, the steps do not naturally emerge in students' minds. Therefore, direct teaching with activities related to practising and drilling construct a large part of students' learning process. However, if the learning steps are identified through an epistemological analysis of the content, the learning steps will present a progressive process of knowledge construction (see Table 3.1 as an example). Given this, one important question needs to be considered; i.e., are teachers in special schools capable of performing an epistemological analysis of mathematics content they are teaching?

In all three cases, teachers and subject advisors demonstrated very different levels of understanding in mathematics. For designing the learning activities of counting, while the teachers were satisfied with using rigid patterns for students memorising the corresponding quantity of a number, the subject advisor knew that the figure went against the abstraction principle of numbers. For the topic of cardinal direction, while the teachers defined cardinal directions by relative directions and sourced students' failure in telling cardinal directions to their left and right confusion, the subject advisor explained that the cardinal direction system and relative direction system are two independent systems and explained why it is problematic to define cardinal directions by relative directions. For teaching mental objects to students with severe ID, while the teachers used different manipulatives to inform student about mental objects (e.g., "being less," "being heavy") and found that the students misunderstood what they meant, subject advisors used the dichotomic nature of the POO activity to distinguish a mental object from others. These three cases reflect the difference in mathematics competency between teachers and subject advisors.

Ma (1999) conducted a comparison research discussing the different understandings of mathematics between mathematics teachers in China and in the United States. In her work, she elaborated on the "profound understanding of fundamental mathematics":

> "Profound understanding of fundamental mathematics (PUFM) is more than a sound conceptual understanding of elementary mathematics—it is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instill those basic attitudes in students. A profound understanding of mathematics has breadth, depth, and thoroughness. Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. The depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics (Ma, 1999, p. 124)."

The framework of POO cannot be developed or applied without a structural understanding of the dichotomic nature of the activity to distinguish a mental object from the otherwise. Without a thorough understanding of numbers, the designed learning activities are fragmented. Without a thorough understanding of direction systems, it is impossible to determine that the definition of cardinal directions suggested in mainstream school mathematics textbooks is problematic. The three cases reflect that a profound understanding of fundamental mathematics is the key to improving mathematics teaching in special schools.

Unfortunately, as stated in Chapter 2, the field of special education rarely provides teachers with the knowledge that teachers badly need. The role of mathematics subject knowledge in teaching in special education is also rarely discussed in the academic field of mathematics special education (Bagger & Roos, 2015; Boyd & Bargerhuff, 2009; Kroesbergen & Van Luit, 2003). Does this mean that more credits focused on mathematics work should be added to the preparation of a mathematics teacher? The specialised content knowledge needed by teachers in this study has shown that this kind of knowledge is not the kind of mathematics knowledge that has been traditionally taught in advanced mathematics courses in institutions and universities. Rather, this is a serious shortcoming of the conceptual underpinnings of the design of mathematics teacher education curriculum.

6.3.2 Implications

In this study, it is found that teachers face teaching problems in which their content knowledge of mathematics is not enough to address the issues and provide a solution. Although most teachers know a great deal of pedagogical knowledge about teaching ID students, breaking a learning task into small parts, providing multisensory activities, etc., they rarely possess the requisite knowledge or skills to assist students in taking the initiative to understand mathematics.

In Hong Kong, there is little subject-specific teacher training for special education. This situation is normally rationalized by the assumption that such teachers can acquire subject-specific preparation from teachers in mainstream education. Although subject-specific preparation for teachers in mainstream education might offer the teacher a basic understanding of the subject, we must realise that there is a set of specialised content knowledge that is only of interest to and badly needed by the professions who teach mathematics in special education. The findings of this study provide considerable evidence on the existence of a set of specialised content knowledge that is unique to teaching ID students, for example, how to solve division without an understanding of multiplication, what the components of counting numbers are, the dichotomic nature of the activity to distinguish a mental object from the otherwise. This result suggest that the knowledge set for teaching elementary mathematics in mainstream schools is not enough for teachers who teach mathematics to students with SENs, especially those with significant disabilities.

This difference in the knowledge sets of teachers in special education and teachers in mainstream education should be taken into consideration in teacher training courses or programmes. The failure of teachers' designs in the three cases described herein was evident proof that mathematics expertise, complemented by some general psychological and pedagogical principles, is not enough to promote effective instructions for teaching mathematics for all. To truly understand the failure, it is necessary to accept the idea that other competences, other forms of knowledge, have to be developed. One has to also consider that if teachers are expected to teach mathematically, specific research to better understand the mathematics teacher profession and knowledge is required.

The BE MATHS programme supports teachers' weakness of mathematics content knowledge by involving subject advisors in the design of teaching and learning trajectories. The team of designers comprises subject experts who have a profound understanding of mathematics and experienced teachers who possess adequate knowledge of teaching and students; having such a combination of members on the team ensures that the design of instruction strictly adheres to the underlying mathematical principle and aligns with students' capability. The role of a subject advisor is to support teachers in analysing the epistemological structure of subject content and initiate the development progress of subject knowledge with constraints of ID. Some teachers reported that collaboration with the subject advisor engaged them in detailed and in-depth discussions about teaching content, which means that teachers can gain valuable specialised content knowledge from the collaboration work with subject experts.

Understanding the knowledge that teachers need to work with ID students can help guide in-service teacher education. To ensure the equity of mathematics education for all students, in addition to pedagogical knowledge of catering to SEN students, this study found that mathematics teachers must acquire specialised content knowledge for teaching mathematics to ID students. As mentioned in Chapter 2, elementary mathematics teachers have a disconnected and rule-based view of mathematics, which influences their teaching proficiency. Therefore, professional development programmes should provide opportunities for teachers studying subject knowledge, especially specialised content knowledge for the mathematics teaching of ID students.

6.4 Limitation and Suggestions for Future Research

The current study focused on a subject-based professional development programme aimed at improving teaching in special education settings for ID students. While in-depth case studies with several in-service teachers revealed meaningful aspects of teacher training, limitations are inherent, given the small subject pool in one type of educational setting.

Future research should be conducted on teachers in other settings (e.g., inclusive education, special education for students with other learning disabilities, and natural science or language subject education). Different results may yield due to these contextual differences.

An interesting area for future research is how the knowledge of teaching accumulated in special education could be transferred to inclusive education. As the trend of inclusive education becomes more popular, teachers in mainstream education will face the challenge of including students with diverse learning needs under their instruction in the same classroom. The experiences and knowledge gained in the ID student group could therefore provide a knowledge base for inclusive education teachers to refine their own teaching.

Although this study found that teachers in BE MATHS programme gained higher scores in all the five CTPD subscales than teachers in the control group, the high correlation between the five scores denotes one important limitation. It is hard to interpret the regression model. To what extent the five programme features contributed to the changes of teachers and students remains open. Therefore, this study may lack some generalisability with regard to the application of findings to in-service teacher education. It is suggested that future research compares the effects of programmes with different features and studies how these programmes' activities contributed to the effects.

Considering that students with ID have a severe delay in language development, this research measured students' academic engaged times to evaluate students' learning achievements. As the academic field knows little about how students with ID learn mathematics, future research is suggested to analyse classroom interactions between students and teachers and record how students' learning statements have been changed over an extended period of time.

In this study, student performances are analysed as evidence of teachers' change. However, to what extent students with various levels of ID can receive mathematics education is a very important study that has not been well done in the current study as well as other studies in mathematics education. Therefore, future research is suggested to analyse and compare the mathematics learning process of students with different ID levels. Individual student's learning processes will be recorded in detail so that the similarities and differences between different ID level students can be compared and summarised.

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Appendix A: Protocol for Mathematics Lesson Observations

Class duration: ______ mins

Read teaching plans and related teaching and learning materials:

- How does the teacher plan the lesson?
- What teaching and learning aids are used?
- What subject content knowledge is reflected in the plans and materials?
- What pedagogical content knowledge is reflected in the plans and materials?
- What pedagogical knowledge is reflected in the plans and materials?

Describe the class setting:

- Student seat plan
- Classroom environment

Observe teachers' performance:

- How does the teacher implement the lesson?
- What subject content knowledge is reflected during the implementation?
- What pedagogical content knowledge is reflected during the implementation?
- What pedagogical knowledge is reflected during the implementation?

Observe students' performance:

- What and how do the students learn in the class?
- Do the students perform as expected by the teacher?
- Are the learning tasks/contents addressing students' learning needs (learning styles, disabilities, background knowledge)?

Observer's reflections:

Appendix B: Protocol for BE MATHS Programme Meeting Observations

Meeting duration: ______ mins
Describe the interactions in the meeting: - Who are the people present? - What are they doing? - What are they talking about? - How did they appear to be interacting with one another? - What subject content knowledge related information are discussed? - What pedagogical content knowledge related information are discussed? - What pedagogical knowledge related information are discussed?
Observe teachers' changes: - What does the teacher plan to teach a given mathematics topic? - Which part does the advisors or other participants agree with the teacher? - Which part does the advisors or other participants disagree with the teacher? - How does the teacher/advisor/other participants adjust their thinking after a discussion about the disagreements? - What changes does the teacher make to his/her instructional plan?
Observer's reflections:

Appendix C: Protocol for Teacher Interviews

Interview duration: ______ mins

What changes do teachers reflect in teaching mathematics to ID students in the BE MATHS programme?

- Before participating in the BE MATHS programme, how did you prepare your mathematics teaching?
- What changes do you reflect now when you prepare your mathematics teaching after participating in the BE MATHS programme?
- Before participating in the BE MATHS programme, how was your teaching? Were you satisfied with your teaching?
- After participating in the BE MATHS programme, how is your teaching? Are you satisfied with your teaching?
- Comparing the mathematics classes that you conducted before and currently conduct after participating in the BE MATHS programme, what skills and knowledge have you developed?
- What changes do you reflect when you teach mathematics to ID students after participating in the BE MATHS programme?
- Before participating in the BE MATHS programme, how did you evaluate your students' learning ability and their learning performance? Were you satisfied with the students' learning outcomes?
- After participating in the BE MATHS programme, how do you evaluate your students' learning ability and their learning performance? Are you satisfied with the students' learning outcomes?

How does the programme influence their mathematics teaching practice and students' performance?

- When you think of the learning experience in the BE MATHS programme for teaching ID students mathematics, what comes first to your mind?
- During the process of teaching ID students mathematics, have you encountered any problems or obstacles? Do the problems and obstacles still exist? If not, how did you solve them? What information/teaching resources/people can help you to solve these problems?
- During the BE MATHS programme meeting, what kind of information are you seeking? Why do you think that kind of information is important to your teaching? Can the BE MATHS programme help you to gain that kind of information?
- What activities of the BE MATHS programme did you attend in the past year? Do you think these activities were meaningful to you? Why?
- In your perspective, after attending the BE MATHS programme, do you have a deeper understanding about mathematics? If yes, what is it? And how did the BE MATHS help you gain that understanding?
- In your perspective, has the BE MATHS programme enhanced your mathematics teaching? If yes, how did the BE MATHS programme help you to do that?
- Do the content and suggestions (instructional designs, teaching materials) provided by the BE MATHS programme align with your school's curriculum and students' learning background? Do you think they are helpful? If yes, in what aspects?

- Have you collaborated with your colleagues/programme advisors/other participants to improve student learning? What have you done? Is the collaboration experience helpful? Why?

Explore a teacher's understanding of a given topic (the topic they or their colleagues have studied)

- When you are teaching this topic to your students, how will you design the topic?
- If one of your students makes the following mistake (show the mistake), how will you interpret the students' thinking? What is the problem behind the mistake? How do you deal with this problem?

Interviewer's reflections: